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The present paper deals with certain classes of generalized vector topologies which in the investigations on general extremality theory in [2] are of importance. For this, let R be a linear space and $\mathfrak{W}_1$ the set of all (T_1-)vector topologies on R. Let $\mathfrak{W}_2$ denote the set of all translation invariant T_1-topologies on R, each of which has an open base B at 0 consisting of equilibrated and absorbing sets such that $\alpha U \in B$ for every $\alpha > 0$ and every $U \in B$. Moreover, let $\mathfrak{W}_3$ denote the set of all translation invariant T_1-topologies on R, each of which has an open base B at 0 consisting of algebraically open sets such that $\alpha U \in B$ for every $\alpha > 0$ and every $U \in B$.

Evidently, $\mathfrak{W}_1 \subseteq \mathfrak{W}_2 \subseteq \mathfrak{W}_3$. The equality signs hold if and only if dim R $\leq$ 1 (see [1], Theorem 13, and [2], Theorem 5). In what follows, we give examples of topologies of $\mathfrak{W}_2 \setminus \mathfrak{W}_1$ and of $\mathfrak{W}_3 \setminus \mathfrak{W}_2$. Let be dim R = 2 and $\{e_1, e_2\}$ a base of R. By B we denote the set of all sets

$$U_{\varepsilon_1,\varepsilon_2} = \Big\{ p = \sum_{i=1}^{2} \pi_i e_i \,\Big|\, |\pi_1| < \varepsilon_1 \text{ and } (\pi_1,\pi_2) \notin \{0\} \times ((-\infty,-\varepsilon_2] \cup [\varepsilon_2,\infty)) \Big\},$$

where ε_1, ε_2 are positive real numbers, respectively the set of all algebraically open sets

$$U_{\varepsilon, p_1, p_2, \ldots} = \Big\{ p = \sum_{i=1}^{2} \pi_i e_i \,\Big|\, \sum_{i=1}^{2} \pi_i^2 < 1;\ p \neq p_i,\ i = 1,2,\ldots \Big\},$$

where ε is a positive real number and p_1, p_2, ... a suitable sequence of points $\neq 0$ converging relative to the natural topology of R to 0. In both cases there exists a unique translation invariant topology T on R with B as open base at 0. In the first case we have $T \in \mathfrak{W}_2 \setminus \mathfrak{W}_1$ and in the second case we have $T \in \mathfrak{W}_3 \setminus \mathfrak{W}_2$.

As is well known, a topology $T \in \mathfrak{W}_2$ belongs to $\mathfrak{W}_1$ if and only if for every $U \in B$ there exists a $V \in B$ with $V + V \subseteq U$. A topology $T \in \mathfrak{W}_3$ belongs to $\mathfrak{W}_2$ if and only if there exists an open base B at 0 consisting of equilibrated sets.

The topologies of $\mathfrak{W}_2$ and $\mathfrak{W}_3$ may be characterized by continuity properties of the vector addition and the scalar multiplication in an analogous way as it is well known for vector topologies.

Theorem 1. A T_1-topology T on R belongs to $\mathfrak{W}_2$ if and only if the following two conditions are satisfied:

i. For every $q \in R$ the mapping $p \to p + q$ of R into R is continuous on R.

ii. The mapping $(\alpha,p) \to \alpha p$ of $\mathbb{R} \times R$ into R is continuous at every point $(\alpha,0)$ and at every point $(0,p)$.

A T_1-topology T on R belongs to $\mathfrak{W}_3$ if and only if the condition i and the following condition are satisfied:

ii'. For every $\alpha > 0$ the mapping $p \to \alpha p$ of R into R is continuous at $p = 0$; for every $p \in R$ the mapping $\alpha \to \alpha p$ of $\mathbb{R}$ into R is continuous at $\alpha = 0$.

Proof. Concerning the characterization of the topologies of $\mathfrak{W}_2$, we refer to [1], Theorem 7. Now we show that for every T_1-topology T on R to belong to $\mathfrak{W}_3$ the conditions i and ii' are necessary and sufficient. At first, let be $T \in \mathfrak{W}_3$. From the translation invariance of T, for arbitrary $q \in R$ we get the continuity of the mapping $p \to p + q$ of R into R on R. Since for every $\alpha > 0$ and every neighbourhood U of 0 the set $\frac{1}{\alpha}U$ also is a neighbourhood of 0, the continuity of the mapping $p \to \alpha p$ of R into R at $p = 0$ is true. The open sets being algebraically open, for every $p \in R$ the mapping $\alpha \to \alpha p$ of $\mathbb{R}$ into R is continuous at $\alpha = 0$. Therefore the conditions i and ii' are satisfied. Conversely, now let T be an arbitrary T_1-topology fulfilling i and ii'. Using condition i, easily we get the translation invariance of T. Let B denote the set of all open neighbourhoods of 0. By the second statement of condition ii' and the translation invariance of T, the sets of B turn out to be algebraically open. From the first statement of condition ii' and the translation invariance of T, it follows $\alpha U \in B$ for every $\alpha > 0$ and $U \in B$. Thus, we have $T \in \mathfrak{W}_3$ and the proof of the Theorem is complete.

Theorem 2. A topology $T \in \mathfrak{W}_2$ belongs to $\mathfrak{W}_1$ if and only if the mapping $(p,q) \to p + q$ of $R \times R$ into R is continuous at $(0,0)$. A topology $T \in \mathfrak{W}_3$ belongs to $\mathfrak{W}_2$ if and only if the mapping $(\alpha,p) \to \alpha p$ of $\mathbb{R} \times R$ into R is continuous at $(0,0)$.

Proof. The first statement of the Theorem is evident. By Theorem 1, also it is obvious that for $T \in \mathfrak{W}_3$ to belong to $\mathfrak{W}_2$ the continuity property of the second statement of the Theorem is necessary. We now prove the sufficiency. Thus, we assume that for a given $T \in \mathfrak{A}_3$ the continuity property is fulfilled. For any neighbourhood U of 0 there exist a $\beta > 0$ and a neighbourhood V of 0 with $(-\beta,\beta)V \subseteq U$. From this, for any $\alpha > 0$ we get $(0,2\alpha)(\frac{\beta}{2\alpha} V) \subseteq U$ and hence we have the continuity of the mapping $(\alpha,p) \to \alpha p$ of $\mathbb{R} \times R$ into R at the point $(\alpha,0)$. For any $p \in R$ let γ be a real number > 0 with $\gamma p \in V$. From $(-\beta,\beta)V \subseteq U$, we get $(-\beta\gamma,\beta\gamma)(\frac{1}{\gamma}(V - \gamma p) + p) \subseteq U$ and hence we have the continuity of the mapping $(\alpha,p) \to \alpha p$ of $\mathbb{R} \times R$ into R at $(0,p)$. By Theorem 1, $T \in \mathfrak{A}_2$. Thus, the Theorem is true.

For any non-empty set $\mathfrak{M}$ of topologies on R, by $T^{\mathfrak{M}}$ we denote the coarsest topology on R which is finer than all topologies of $\mathfrak{M}$. As is well known, $\mathfrak{M} \subseteq \mathfrak{A}_1$ yields $T^{\mathfrak{M}} \in \mathfrak{A}_1$. Analogous statements with regard to $\mathfrak{A}_2$ and $\mathfrak{A}_3$ also are true, that is, we have

<u>Theorem 3</u>. For $\mathfrak{M}(\neq \emptyset) \subseteq \mathfrak{A}_i$ $(i = 2,3)$, $T^{\mathfrak{M}} \in \mathfrak{A}_i$.

<u>Proof</u>. [1], Theorem 11, and [2], Theorem 4.

As for $i = 1$, because of Theorem 3, for $i = 2, 3$ in $\mathfrak{A}_i$ there exists a finest topology $T^{\mathfrak{A}_i}$. Concerning a characterization of $T^{\mathfrak{A}_1}$, we refer to [3], 6.C. By [1], Theorem 12, and [2], Theorem 5, we get the following

<u>Theorem 4</u>. $T^{\mathfrak{A}_2}$ is the topology on R which has as an open base at 0 the set of all subsets U of R such that for every finite-dimensional subspace R' of R relative to the natural topology of R' the set $U \cap R'$ is an equilibrated open neighbourhood of 0. $T^{\mathfrak{A}_3}$ consists of all algebraically open sets of R.

<u>Theorem 5</u>. $T^{\mathfrak{A}_2}$ belongs to $\mathfrak{A}_1$ if and only if dim R is finite. $T^{\mathfrak{A}_3}$ belongs to $\mathfrak{A}_2$ (and hence to $\mathfrak{A}_1$) if and only if $\dim R \leq 1$.

<u>Proof</u>. [1], Theorem 12, and [2], Theorem 5.

With regard to the partial ordering $\leq$ given by $T \leq T' \Leftrightarrow T \subseteq T'$, the topologies $T^{\mathfrak{A}_2}$ and $T^{\mathfrak{A}_3}$ are the maximal elements of $\mathfrak{A}_2$ and $\mathfrak{A}_3$, respectively. As to minimal elements of $\mathfrak{A}_2$ and of $\mathfrak{A}_3$, we have the following

<u>Theorem 6</u>. Let be $\dim R \geq 2$. Then the minimal elements of $\mathfrak{A}_2$

and of $\mathfrak{W}_3$ do not belong to $\mathfrak{W}_1$.

Proof. The statement is an immediate consequence of [1], Theorem 13.

In what follows, we restrict ourselves to the case in which R has a finite dimension $n \geq 2$. Let be $\{e_1, \ldots, e_n\}$ a base of R and μ the euclidean norm on R with respect to this base. Moreover, let be $K = \{p \in R; \mu(p) < 1\}$ and $\partial K = \{p \in R; \mu(p) = 1\}$. For any $p \in \partial K$ we denote by R_p the (n-1)-dimensional linear subspace of R consisting of all points of R orthogonally to p and by π_p the orthogonal projection of R onto R_p. Let T' be the natural topology of R.

Theorem 7. For any $T \in \mathfrak{W}_2$, $T \leq T'$. For $T \in \mathfrak{W}_3$, $T' \leq T$ if and only if for every point $p \in \partial K$ there exist an equilibrated T-open set U with $0 \in U \cap R_p \subseteq K$, a point $q \in U$ with $\mu(\pi_p(q)) > 1$, and an equilibrated T-open set V with $0 \in V \subseteq U \cap (U + q)$.

Proof. Concerning the first statement of Theorem 7, we refer to [1], Theorem 9. In [1], Corollary to Theorem 9, the second statement is proved in the special case $T \in \mathfrak{W}_2$. The proof in the case $T \in \mathfrak{W}_3$ is obtained from this by some slight modifications.

Corollary. For $T \in \mathfrak{W}_2$, $T = T'$ if and only if for every point $p \in \partial K$ there exist a T-open set U with $0 \in U \cap R_p \subseteq K$ and a point $q \in U$ with $\mu(\pi_p(q)) > 1$.

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