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MAXIMAL LINKED SYSTEMS IN TOPOLOGY

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1. Maximal linked systems of closed sets

Ten years ago J. DE GROOT introduced maximal linked systems in topology.

Let X be any set. A collection $L \subset P(X)$ is *linked* if $L_1 \cap L_2 \neq \emptyset$ for all $L_1, L_2 \in L$. If also $S \subset P(X)$, then L is a *maximal linked system in S* (shortly, an S-mls) if L is a linked subsystem of S not properly contained in any other linked subsystem of S. We will write $\mathbb{L}(X,S)$ for the set of all maximal linked subsystems of S.

DE GROOT was concerned with the following situation: X is a T_1-space, S is a subbase for the closed sets of X, and $\mathbb{L}(X,S)$ is topologized by taking

$$S^+ := \{S^+ : S \in S\},$$

with

$$S^+ := \{L : S \in L \in \mathbb{L}(X,S)\} \quad (S \in S),$$

as a subbase for its closed sets. The space obtained in this way he denoted by $\lambda_S(X)$; we will call it $\lambda(X,S)$.

The following notational conventions will simplify our presentation. If X is a set and $S \subset P(X)$, then $[X,S]_{cl}$ and $[X,S]_{op}$ will denote the topological space with X as the underlying set and S as a subbase for the closed sets (for the open sets, respectively). Thus

$$\lambda(X,S) = [\mathbb{L}(X,S), S^+]_{cl}.$$

In comparison, let $\mathbb{F}(X,S)$ stand for the collection of all maximal *centered* subsystems of S; if $S \in S$, let

$$S^{\times} := \{F \,\vdots\, S \in \mathbb{F}(X.S)\}$$

and let

$$S^{\times} := \{S^{\times} \,\vdots\, S \in S\}.$$

Then

$$\omega(X,S) := [\mathbb{F}(X,S),\ S^{\times}]_{cl}$$

is the WALLMAN-type compactification of $X = [X,S]_{cl}$ relative to the closed subbase S. In the special case where S is the collection of all closed subsets of $X = [X,S]_{cl}$, we will omit reference to it and we will just write $\lambda(X)$ and $\omega(X)$.

Just as all WALLMAN-spaces $\omega(X,S)$ are compact, all spaces $\lambda(X,S)$ are compact too: they are even *supercompact*, in the following sense.

DEFINITION. A closed subbase S for a topological space X is called *binary* if every linked subsystem of S is fixed. A topological space X is called *supercompact* if it has a binary closed subbase.

If S is a binary closed subbase for X, then $S^{\star} := \{X\backslash S \,\vdots\, S \in S\}$ is an open subbase for X with the property that every open $S^{\star}$-cover had a 2-element subcover (let us call such an open subbase *open-binary*). From ALEXANDER's lemma it is obvious that every supercompact space is compact.

DE GROOT called the space. $\lambda(X,S)$ the *superextension of* X *relative* S (*the* superextension of X being the space $\lambda(X)$). However, $\lambda(X,S)$ rarely is an extension of X, in the sense that (a copy of) X is dense in $\lambda(X,S)$ (cf. [17] section 7). But $\lambda(X,S)$ is a superspace of X if X is a T_1-space and S a T_1-subbase, in the following sense.

DEFINITION. A closed subbase S for a topological space X is called a

T_1-subbase it $(\forall x \in X)(\forall S \in S)(x \notin S \Rightarrow (\exists T \in S)(x \in T \subset X\backslash S))$;

weakly normal subbase if $(\forall S_1, S_2 \in S)(S_1 \cap S_2 = \emptyset \Rightarrow$ there are finitely many $T_1,\ldots,T_n \in S$ such that

each T_i meets at most one of S_1 and S_2
and $T_1 \cup \ldots \cup T_n = X$);

normal subbase if $(\forall S_1, S_2 \in \mathcal{S})(S_1 \cap S_2 = \emptyset \Rightarrow (\exists T_1, T_2 \in \mathcal{S})(T_1 \cup T_2 = X$ and $T_1 \cap S_1 = T_2 \cap S_2 = \emptyset))$.

Introduce the following map i, defined on X (and depending on $\mathcal{S}$):

$$i(x) = \{S \,\vdots\, x \in S \in \mathcal{S}\}.$$

Then the following holds.

THEOREM 1. *If* X *is a* T_1*-space and* $\mathcal{S}$ *a* T_1*-subbase for the closed sets of* X, *then* i *is a topological embedding of* X *in* $\lambda(X,\mathcal{S})$.

Let $\beta(X,\mathcal{S})$ denote the closure of $i[X]$ in $\lambda(X,\mathcal{S})$; under the assumptions of theorem 1, $\beta(X,\mathcal{S})$ is a compactification of X. In [14] and in later papers these compactifications were called *GA-compactifications*. They were used by DE GROOT and AARTS [11] to obtain their well-known *subbase* characterization of complete regularity.

THEOREM 2. *Let* $X = [X,\mathcal{S}]_{cl}$ *be a* T_1*-space, with* $\mathcal{S}$ *a weakly normal* T_1*-subbase for the closed sets. Then* $\beta(X,\mathcal{S})$ *is a Hausdorff compactification of* X; *it is a quotient of the WALLMAN-type compactification* $\omega(X,\mathcal{S})$ *(which under the given assumptions need not be* T_2*).*

COROLLARY. *A* T_1*-space* X *is completely regular if and only if it has a weakly normal* T_1*-subbase for the closed sets.*

Recently VAN MILL [18] proved that every Hausdorff compactification of a locally compact separable space is (equivalent to) a GA-compactification. Whether this also holds for general spaces is an open problem that seems far from being solved.

Many nice results on maximal linked systems of closed sets, and a good bibliography up to 1972, are to be found in VERBEEK's monograph [26]. A survey of recent results is under preparation [2]. Let me just mention a very few that to me seem important and beautiful.

STROK and SZYMAŃSKI proved that *every compact metric space is supercompact* [24]. Other classes of supercompact spaces are: *all compact orderable spaces, all compact treelike spaces* ([5],[17]), *all compact lattice spaces* [23],[21]. As super compactness is preserved under the formation of products, there are many supercompact spaces indeed. However, since recently it is known also that there are many *non*-supercompact compact Hausdorff spaces. First, BELL [4] showed that *if* X *is not pseudocompact then* βX *is not supercompact*; hence e.g. $\beta\mathbb{N}$ is not supercompact. Next VAN DOUWEN and VAN MILL [7] considerably strengthened this result; they proved:

THEOREM 3. *Let* X *be a* T_2*- space which is a continuous image of a supercompact* T_2 *space. If* K *is any countably infinite subset of* X*, then*

(a) *at least one cluster point of* K *is the limit of a non-trivial convergent sequence in* X *(not necessarily in* K*), and*
(b) *at most countably many cluster points of* K *are not the limit of some non-trivial convergent sequence in* X.

As a consequence, no infinite extremely disconnected space or, more generally, no infinite Hausdorff F-space can be (a continuous image of) a supercompact space.

The result we are going to mention next solves a long-standing conjecture of DE GROOT; it is simple enough to formulate but the proof is highly technical and very ingenious: *the superextension of the unit segment is the Hilbert cube* (VAN MILL [19]).

Another recent result about superextensions, by VAN DE VEL [25], states that *the superextension of a connected normal* T_1*-space always has the fixed point property.*

Just before his untimely death DE GROOT was working on an interesting connection between supercompact spaces and (infinite) graphs (see [10] and [6]). His ideas in this direction were taken up and succesfully extended by A. SCHRIJVER [23], [21]).

2. Maximal linked systems of open sets

In this section we assume throughout that S is an *open* subbase for a *Hausdorff* space X.

DEFINITION. A linked system L of subsets of X is *convergent* if

$$|\cap\{\overline{L} : L \in L\}| = 1;$$

if this is the case, and $\cap\{\overline{L} : L \in L\} = \{p\}$, then p is called the limit of L; notation: $p = lim\ L$.

A linked system of closed sets converges if and only if it is a convergent *centered* system; therefore the theory of convergent maximal linked systems of closed sets is contained in the theory of fixed maximal centered systems. For mls's of open sets the situation turns out to be radically different.

We need some additional notation. Suppose $X = [X,S]_{op}$; then we put

$$\Lambda(X,S) := [\mathbb{L}(X,S), S^{+}]_{op}$$

and

$$\Omega(X,S) := [\mathbb{F}(X,S), S^{\times}]_{op};$$

the subbase of $\Lambda(X,S)$ consisting of all convergent S-mls's is denoted $B(X,S)$, and the subspace of $\Omega(X,S)$ consisting of all convergent maximal centered S-systems is denoted $A(X,S)$.

Also, we need some additional properties for subbases.

DEFINITION. An open subbase S for a space X is called a

complemented subbase if $(\forall S \in S)(\exists S' \in S)(S \cap S' = \emptyset \wedge (\forall T \in S)(T \cap (S \cup S') \neq \emptyset))$.

Although some of the results which follow are valid in a more general setting, we will assume for simplicity's sake that S is a *ring* of sets. Hence S is an open base, and

$$\Omega(X,S) \subset \lambda(X,S); \qquad A(X,S) \subset B(X,S).$$

THEOREM 4. *Let* $X = [X,S]_{op}$ *be a* T_2*-space, where* S *is a ring of sets.*

(i) $\Lambda(X,S)$ *is a zero-dimensional* T_2*-space, with density at most* $dX + \omega$;

(ii) *lim*: $B(X) \to X$ *is a* θ*-continuous surjection*;

(iii) *if in addition* S *is complemented, then* $\Lambda(X,S)$ *is supercompact; in fact,* $\Lambda(X,S)$ *is a superextension both of* $A(X,S)$ *and of* $B(X,S)$,

(iv) *if* S *is complemented and* X *is compact, then* $B(X,S)$ *is compact (but in general not supercompact)*;

(v) *if* S *is complemented, then lim:* $B(X,S) \to X$ *is perfect; point-originals under lim are even supercompact.*

In the special case where S is the collection of all open sets of $[X,S]_{op}$, we will suppress S in the notation, writing $\Omega(X)$, $A(X)$, $\Lambda(X)$, $B(X)$ etc. In this case $A(X)$ is the *absolute* of X, and $\Omega(X)$ is what ILIADIS [15] calls the *hyperabsolute* of X (cf. also [16] and [22]; in case X is completely regular, $\Omega(X)$ is just $A(\beta X)$).

THEOREM 5. *The following assertions are equivalent in case* X *is regular:*

(i) $B(X)$ *is extremely disconnected*;

(ii) $B(X) = A(X)$;

(iii) *lim*: $B(X) \to X$ *is a homeomorphism*;

(iv) X *is extremely disconnected*;

(v) $B(X)$ *is an F-space*;

(vi) $B(X)$ *contains no convergent sequences.*

THEOREM 6. *Let* S *be a normal* T_1 *subbase for the closed sets of* X. *Then the map lim:* $B(X) \to X$ *has a continuous extension* ℓ: $\Lambda(X) \to \lambda(X,S)$ *such that* ℓ *is perfect and surjective, while in addition* $\ell\,[\Omega(X)] = \beta(X,S)$. *If in addition* S *contains all open-and-closed subsets of* X, *then the described extension* ℓ *is injective (and hence a homeomorphism) if and only if* X *is extremely disconnected.*

It now easily follows that $B(AX)$, is homeomorphic to $A(X)$, and also that the spaces $\Lambda(X)$, $\Lambda(\beta X)$, $\Lambda(AX)$ and $\Lambda(\Omega X)$ are all homeomorphic. In addition, ΛX is homeomorphic to *the* superextension $\lambda(\Omega X)$ of the hyperabsolute of X, and to the zeroset superextension $\lambda_0(AX)$ of the absolute of X (where

$\lambda_0(Y)$ stands for the superextension $\lambda(Y,\mathcal{Z}(Y))$.

As supercompact spaces cannot be extremally disconnected (except if they are finite), the space $\Lambda(X)$ (with X infinite) is never extremally disconnected. It is a retract of a space $\Lambda(D)$ with D discrete.

ILIADIS [15] has shown that any irreducible θ-continuous closed surjection $f\colon X \to Y$ can be lifted to a homeomorphism $\Omega f\colon \Omega(X) \to \Omega(Y)$ such that $(\Omega f)[A(X)] \subset A(Y)$. We have proved that f can even be lifted to a homeomorphism $\Lambda f\colon \Lambda(X) \to \Lambda(Y)$ such that $(\Lambda f) \mid \Omega(X) = \Omega f$, and such that always $(\Lambda f)[B(X)] \subset B(Y)$. In addition it turns out that $(\Lambda f)[B(X)] = B(Y)$ if and only if f itself is a homeomorphism.

Finally we mention - without going into details - that B(X) can be desribed as a maximal pre-image of X under a special kind of mappings. A paper on $\Lambda(X,\mathcal{S})$ and $B(X,\mathcal{S})$ and some related spaces spaces, by J. VAN MILL, E. WATTEL and the author, is under preparation; it will appear as a report of the Mathematical Centre in Amsterdam [3].

The results in this section were obtained in pleasant coöperation with Jan VAN MILL and Evert WATTEL.

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