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SOME QUESTIONS RELATED TO HYPERSPACES

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All spaces under consideration are supposed to be compact and Hausdorff. For a space X, exp(X) denotes the set of all non-empty closed subsets of X taken with the finite topology. For two spaces X and Y, X + Y denotes their disjoint topological sum ($2X = X + X$) and X x Y their topological product ($X^2 = X \times X$).

Although the spaces exp(X) have been intensively investigated more than fifty years, not many concrete examples (which can be identified as subspaces of coordinate spaces) were known untill recently. One could think of the trivial examples of finite spaces having $2^n - 1$ points (and let $2^n - 1$ denote such a space), plus the Cantor set C and by the application of

$$\exp(X + Y) \approx \exp(X) + \exp(Y) + \exp(X) \times \exp(Y),$$

the spaces $(2^n - 1) + C$ can be added.

In 1972, D. W. Curtis and R. M. Schori [2] proved that exp(X) is the Hilbert cube for each non-degenerated Peano continuum X. In 1965, A. Pelczynski [6] proved that the space T(C) consisting of the Cantor set plus the centers of removed intervals is the hyperspace of all 0-dimensional metric compact spaces having the set of isolated points dense. This result discovered for the first time the possibility of quite different spaces having homeomorphic hyperspaces. In [4] , six further examples were provided which, together with the already mentioned ones, exhaust the list of hyperspaces for the class of 0-dimensional compact metric spaces.

On the basis of the result of Curtis and Schori and the results in [4] and [5] followed by a simple verification, an immediate conclusion is that for any two spaces X and Y which are either Peano continua or compact metric and 0-dimensional:

1. $X^2 \approx Y^2$ implies $\exp(X) \approx \exp(Y)$.

One could be motivated to conjecture the validity of 1. in general case of compact Hausdorff spaces.

The implication under 1. can be "factorized" into the following two:

2. $X^2 \approx Y^2$ implies $X^{[2]} \approx Y^{[2]}$,

3. $X^{[2]} \approx Y^{[2]}$ implies $\exp(X) \approx \exp(Y)$,

where $X^{[2]}$ stands for the symmetric square of the space X and can be thought of as the subspace of exp(X) consisting of all subsets of cardinality at most two. Two spaces X^2 and $X^{[2]}$ often differ (case of a finite X, case $X = S^1$ [1] and several other cases) but they have many topological properties in common. The question of validity of the implication under 3. in case of compact Hausdorff spaces seems technically easier to attack than the one under 1.

For both Peano continua and the class of compact metric and 0-dimensional spaces the following implication

4. $X \approx \exp(X)$ implies $X \approx X^2$

holds. May the implication under 4. be true in case of an arbitrary compact Hausdorff space?

Since the natural numbers can be identified with the finite spaces and the two operations with the corresponding topological operations on spaces, the topological types of class of compact metric 0-dimensional spaces (the class being denoted by $\mathfrak{Z}$) become a close generalization of the system of natural numbers. There are several questions which can be asked on this basis and whose solutions would contribute in part to a better understanding of the class $\mathfrak{Z}$ (and, may-be of natural numbers). Let us restrict ourselves to the following two.

Only nine hyperspaces in $\mathfrak{Z}$ have the property

5. $X^2 \approx X$

Are there some other solutions in $\mathfrak{Z}$ of the equation under 5. (is what I do not know)?

(One could set his mind on solving the equation under 5. or the equation of the form $X \approx 2X$, but without having a clear idea what a description of the set of solutions should be I restrain myself from such possibly bombastic formulations).

Now we formulate our last question.

6. Could there exist a space X in $\mathfrak{Z}$ such that

$$X \not\approx 2X \text{ and } 2X \approx 3X \approx \ldots$$

As for the order relation on topological types in $\mathfrak{Z}$, it would have been natural to put $X \leq Y$ if X is homeomorphic to a closed and open subspace of Y but an example from [3] (p. 126) shows it be not feasible (for the whole $\mathfrak{Z}$).

The formulation of these six questions is normally such that the positive answers would give signs of my present conviction and whatever they are, I hope, they could be of some interest.

References

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