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In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [329]--330.

Persistent URL: http://dml.cz/dmlcz/700631

SUBCATEGORIES WITH CARTESIAN CLOSED COREFLECTIVE HULL

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It has long been known that by forming all quotients of direct sums of compact Hausdorff spaces in the category of topological spaces, one obtains a cartesian closed topological category. Recently G. Tashjian revealed that precompact seperated spaces likewise have a coreflective hull in the category of seperated uniform spaces which is cartesian closed. The question arises: what conditions must a subcategory satisfy in order to have a cartesian closed coreflective hull? We give an answer to this which unifies known special cases and provides a tool for finding new ones.

We use "topological category" in the sense of H. Herrlich: a concrete category with initial structures, small fibres and precisely one structure on a single point set. Typical examples from a vast collection are the usual categories of topological spaces, uniform spaces, regular topological spaces, competely regular topological spaces, nearness spaces, semi-nearness spaces (seperation axioms are never implied in this paper unless stated). A topological category C is cartesian closed iff each hom-set $C(X,Y)$ can be structured to become a function space $C[X,Y]$ so that the exponential law $C[X\times Y,Z] \cong C[X,C[Y,Z]]$ holds; the natural isomorphism carries f to f* where $f^*(x)(y) = f(x,y)$.

Cartesian closed topological categories are remarkably powerful and elegant in theories where function spaces play a crucial role e.g. in certain areas of functional analysis. The talks by E. Binz and H.E. Porst given at this symposium provide excellent illustrations. The topological categories mentioned above are each of considerable interest, but none are cartesian closed. This quickens interest in the formation of subcategories of them which do have this desirable property and are at the same time topological categories. Subcategories are full and isomorphism-closed.

Let us henceforth suppose that X is a given topological category and K a subcategory closed under finite products. Our main result can be stated as follows.

THEOREM: *The coreflective hull of* K *in* X *is a cartesian closed topological category provided that for any objects* K,L *in* K *and* Y *in* X *there exist a function space object* $X[K,Y]$ *in* X *such that the assignment* $f \rightsquigarrow f^*$ *is a bijection of* $X(K\times L,Y)$ *to* $X(L,X[K,Y])$

Usually $X[K,Y]$ does not itself belong to the coreflective hull of K: it serves only as a preliminary function space from which the real thing is derived by coreflec -tion. Let us mention a few situations where this theorem is applicable. In each case we can obtain further examples by taking a finitely productive subcategory of the given K.

EXAMPLES: (a) In the category of topological spaces we take the subcategory of all spaces with neighbourhood filters having a base of compact subsets. The required function spaces are formed with the compact open topology, also in the next two examples.

(b) In the category of regular topological spaces take the subcategory of compact spaces.

(c) In the category of completely regular spaces take the subcategory of compact Hausdorff spaces.

(d) In the category of topological spaces take the subcategory of continuous lattices in the sense of D.S. Scott i.e. T_0-spaces injective with respect to embeddings. Here the function spaces are formed with the topology of pointwise convergence.

(e) In the category of semi-nearness spaces take the subcategory of contigual spaces (they are generalizations of precompact spaces). The function spaces here do not admit brief description, but we note that for uniform spaces their structure reduces to the uniformity of uniform convergence. Thus we obtain the last example as a special case.

(f) In the category of uniform spaces take the subcategory of precompact spaces.

It turns out that the smallest (in a suitable sense) cartesian closed topological category containing all compact Hausdorff spaces is the one obtained by applying our theorem to example (c). Corresponding facts for continuous lattices and precompact spaces result from examples (d) and (f) respectively.

Details and references will appear elsewhere in final form.