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A SURVEY OF RECENT RESULTS ON THE SPECTRAL RADIUS IN BANACH ALGEBRAS

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The results we have to present here*) are concerned with characterizations of commutativity in general Banach algebras by means of properties of the spectral radius. We have obtained both global characterizations as well as local conditions for single elements to belong to the centre or to the radical. They all, the results and their proofs, demonstrate the close connections which link the algebraic, topological and analytic features of the subject. Several open questions are also mentioned.

I. Spectral characterizations of commutativity: global conditions

It follows from Gelfand's theory of commutative normed algebras that the spectrum of elements in such algebras possesses some nice algebraic and continuity properties. In view of the general importance of the notion of spectrum it should be quite desirable to examine, conversely, to what extent these nice properties can affect the commutativity, the essence of the Gelfand theory. It is rather surprising that such characterizations of commutativity can be expressed even in terms of the spectral radius. Indeed, the first principal result reads as follows.

THEOREM 1. Let A be an arbitrary Banach algebra. Then the following four conditions are equivalent:

1^o the spectral radius is subadditive on A ;

2^o the spectral radius is submultiplicative on A ;

3^o the spectral radius is uniformly continuous on A ;

4^o the algebra $A/\text{rad } A$ is commutative.

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Here, we say that the spectral radius is subadditive if there exists a constant α such that $|x + y|_\sigma \leq \alpha(|x|_\sigma + |y|_\sigma)$ for all x,y in A , and submultiplicative if similarly $|xy|_\sigma \leq \beta |x|_\sigma |y|_\sigma$ for all x,y in A with some constant β . By rad A we denote the (Jacobson) radical of A . Certainly any spectral characterization of commutativity should be expected only modulo the radical since, although the spectral radius on a radical algebra - being identically zero - possesses all the nicest properties, no commutativity in the algebra, however, can be inferred.

The simplicity of the proof of Theorem 1 fully corresponds to the simplicity of its statement. Rather curiously we do not have to use any explicit formula for the spectral radius. As the only essential ingredient there appears the Liouville theorem applied (in the step $3^o \rightarrow 4^o$) to the entire function $e^{\lambda a} b e^{-\lambda a}$ which, with respect to the semi-norm

$$q(x) = \sup_{z \in A} (|x + z|_\sigma - |z|_\sigma) , \qquad x \in A ,$$

turns out to be bounded since obviously

$$q(cxc^{-1}) = q(x)$$

for all c regular; it follows from 3^o that $q(x) \leq \gamma |x|$ for all $x \in A$ and a suitable constant γ. Consequently, the Liouville theorem yields $q(ab - ba) = 0$ for all a,b in A . Now, it is a general fact that $q^{-1}(0)$ coincides with the radical in any Banach algebra (cf. Theorem 3 in part II of this lecture; a more thorough study of the function q will be the object of part III). However, under the assumptions of Theorem 1 this fact can be established immediately. Therefore the rest of the proof of Theorem 1 consists in studying relations between different forms of subadditivity and submultiplicativity of the spectral radius which is a purely algebraic matter and which, by the way, gives all the remaining implications in Theorem 1. In particular, a simple direct proof of the equivalence of conditions 1^o and 2^o is available. At the same time it becomes obvious that certain formally weaker conditions also imply 4^o; we refer the reader to the papers [7] and [4] where complete proofs as well as a history of the problem are to be found. A considerable refinement of all these conditions we shall see in part III where - using more delicate methods - we have obtained a local version of Theorem 1, that means conditions on a single element to belong to the centre of the algebra.

It should be pointed out here that the first spectral characterization of commutativity in general Banach algebras (consisting in the observation that both the conditions 1^o and 2^o together imply 4^o) was implicitly suggested already in the paper [2] by R.A. Hirschfeld and W. Żelazko. Let us also call attention to the just appearing paper [1] by B. Aupetit who has obtained results slightly stronger than our Theorem 1 independently by deeper methods based on the theory of subharmonic functions. Similar questions have been recently considered also by E.A. Gorin who has originally introduced the semi-norm q .

In connection with Theorem 1 one remark is necessary. A characterization of such algebraic property as commutativity modulo the radical should not depend on the (possibly non-equivalent) Banach algebra topologies which the algebra A can share. But condition 3^o of Theorem 1 apparently does depend on the topology. However, this can be explained as follows. First of all, it is well-known and easy to prove that the spectrum of an element in A is the same as the spectrum of the corresponding class in $A/\text{rad}\, A$. Secondly, the algebra $A/\text{rad}\, A$ being semi-simple, its topology is uniquely determined - according to a celebrated theorem of B.E. Johnson - by the algebraic structure of A alone. Therefore continuity or uniform continuity of the spectrum or of the spectral radius on A , which is easily seen to be the same as that on $A/\text{rad}\, A$, is, in fact, an algebraic property. We have just shown that the uniform continuity corresponds exactly to the commutativity of the algebra modulo its radical. It would be interesting to find the algebraic properties which correspond to ordinary continuity of the spectral radius or of the spectrum.

Likewise the equivalence of conditions 1^o and 2^o in Theorem 1 indicates that there should be deeper relations between the additive and multiplicative structures of Banach algebras. Indeed, this analogy will also be followed in the further parts of the lecture.

II. Spectral characterizations of the radical

Next, denote by $N = \{x \in A : |x|_6 = 0\}$ the set of all quasinilpotent elements of the algebra A . It is well-known that $N \supset \text{rad}\, A$, but this inclusion can often be proper. A natural characterization of algebras in which equality occurs is provided by the following

THEOREM 2. Let A be a Banach algebra. Then the following three conditions are equivalent:

1^o $x,y \in N$ implies $x + y \in N$;

2^o $x,y \in N$ implies $xy \in N$;

3^o $N = \text{rad } A$.

Observe that the first two conditions in Theorem 2 are weakenings of the corresponding conditions in Theorem 1. These two weakenings do not imply the commutativity any more, but it is interesting that they are again equivalent. They characterize a class of algebras which lies between commutative and general Banach algebras. (There is an elegant example due to A.S. Nemirovskij or, independently, to J. Duncan and A.W. Tullo which shows that a Banach algebra A with $N = \text{rad } A = 0$ need not be commutative.) It would be interesting to investigate whether the spectral radius is continuous in this class (cf. the problem posed in part I).

Theorem 2 has been first conjectured in the author's paper [7] and later proved in the common paper [5] by Z. Słodkowski, W. Wojtyński and the present author. The equivalence of conditions 1^o and 2^o is, similarly as in Theorem 1, again a purely algebraic matter; to establish it, one can use the following two decompositions:

$$1 - xy = (1 - x)(1 + u + v)(1 - y) ,$$

$$1 - (x + y) = (1 - x)(1 - uv)(1 - y) ,$$

where $u = (1 - x)^{-1}x$ and $v = y(1 - y)^{-1}$ have zero spectra if x,y are in N ; a more detailed quantitative investigation of this type decompositions leads to the proof of the equivalence of conditions 1^o and 2^o in Theorem 1 (cf. [7] and [4]).

However, the proof of the condition 3^o in Theorem 2 is more delicate since we do not know in advance whether the set N is closed. Here the situation is not appropriate for Liouville's theorem. In our proof [5] there are two essential points which deserve to be isolated as independent lemmas. The first one goes back to an idea of C. Le Page [3] consisting in an ingenious application of Jacobson transitivity of strictly irreducible representations. A complete proof can be found in [5]; see also [7] where a similar idea has been applied.

LEMMA 1. Let $r \in N$ be such that $ar - ra \in N$ for all $a \in A$. Then $r \in \text{rad } A$.

It may be worthwhile at this point to compare Lemma 1 with the original result of C. Le Page [3] which we shall need in part III:

LEMMA 2. Let $b \in A$ be such that $ab - ba \in N$ for all $a \in A$. Then $ab - ba \in \text{rad } A$ for all $a \in A$.

The second essential lemma in the paper [5] was originally motivated by a certain result of I.N. Herstein concerning Lie ideals. Later it was reformulated by the author [8] to the following form which seems to be more convenient for further applications.

LEMMA 3. Let $r \in N$ be such that $x + r \in N$ for all $x \in N$. Then $ar - ra \in N$ for all $a \in A$.

The crucial point in the proof of Lemma 3 is the use of the theorem of E. Vesentini [6] on the subharmonicity of the spectral radius to the function

$$\left|(e^{\lambda a} r e^{-\lambda a} - r)/\lambda\right|_{\sigma}$$

at $\lambda = 0$. See [5] for details; in part III a similar idea will be applied in the proof of Theorem 4.

Now, the implication $1^{\circ} \rightarrow 3^{\circ}$ in Theorem 2 is simply a conjuction of Lemmas 3 and 1 in this order. In particular, observe that the purely algebraic conditions 1° or 2° imply the closedness of the set N (since the radical is always closed); on the other hand, if A is, for instance, the algebra of all bounded operators on an infinite-dimensional Hilbert space, then the corresponding set N is not closed (by a well-known example due to S. Kakutani), hence the spectral radius is discontinuous on this important algebra.

Although the Jacobson radical of a Banach algebra is not - in general - simply the kernel of the spectral radius, it nevertheless admits characterizations in terms of the spectral radius. One such characterization is based on the following observation. We have already mentioned (in part I) that $\sigma(a + r) = \sigma(a)$ for all $a \in A$, $r \in \text{rad } A$. In other words, perturbations by elements from the radical leave the spectrum invariant. Now the natural question arises whether, conversely, the condition $\sigma(a + r) = \sigma(a)$ for all $a \in A$ and some $r \in A$ forces r to be in the radical. An affirmative answer to this question follows immediately from Lemmas 3 and 1 quoted above. Moreover, it becomes obvious that it suffices to require the condition not for all $a \in A$ but only for all $a \in N$. So we have

THEOREM 3. Let A be an arbitrary Banach algebra. Suppose $r \in A$ is such that $|a + r|_\sigma = 0$ for all $a \in N$. Then $r \in \operatorname{rad} A$.

COROLLARY 1. Let A be a Banach algebra. Then an element $r \in A$ belongs to the radical if (and only if) $\sigma(a + r) = \sigma(a)$ for all $a \in A$.

Also, it is now clear that $q^{-1}(0) = \operatorname{rad} A$, as we claimed already in part I of this lecture. We shall see this result once more in a wider context of part III. For further corollaries including applications to operator algebras we refer the reader to the papers [8] and [9].

Corollary 1 suggests possibility of a spectral characterization of elements belonging to the primitive ideals. If $r \in A$ belongs to some two-sided ideal, then we have only $\sigma(a + r) \cap \sigma(a) \neq \emptyset$ for all $a \in A$. Is the converse true? An affirmative answer is known in the case of the algebra $B(H)$ - the work of J.A. Dyer, P. Porcelli and M. Rosenfeld, and, of course, is obvious for commutative algebras. A related question is whether a Banach algebra in which every primitive ideal is prime must be commutative (modulo the radical).

The results of part II as well as of the next part III, in contrast to those of part I, do depend on the spectral radius formula

$$|x|_\sigma = \lim |x^n|^{1/n}$$

which is essentially used in the proof of the theorem of E. Vesentini [6].

III. Spectral characterizations of the centre

In Theorem 1 we have given natural global conditions, expressed in terms of the spectral radius, on the whole algebra to be commutative (modulo the radical). In this third part, being suggested by the two preceding parts, we shall find conditions - again purely in terms of the spectral radius - for a single element to belong to the centre. By centre we mean here the centre modulo the radical, i.e. the set

$$Z(A) = \{x \in A : ax - xa \in \operatorname{rad} A \text{ for all } a \in A\} .$$

Clearly, $Z(A)$ is a closed subalgebra containing the radical of A.

THEOREM 4. Let A be an arbitrary Banach algebra. For each $x \in A$ put

$$q(x) = \sup_{y \in A} (|x+y|_\sigma - |y|_\sigma) .$$

Then

1^o $x \in Z(A)$ if and only if $q(x) < \infty$;

2^o $x \in \text{rad } A$ if and only if $q(x) = 0$.

In fact, for each $x \in A$ there are only two possibilities:

either $q(x) = |x|_\sigma$, if $x \in Z(A)$,

or $q(x) = \infty$, if $x \notin Z(A)$.

Moreover,

$$q(x) = \sup_{s} (|x-s|_\sigma - |x|_\sigma)$$

where s runs only the set

$$S(x) = \{gxg^{-1} : g \text{ regular}\} \cup \{-x\} .$$

Of course, condition 1^o applied to all $x \in A$ gives the global result of part I. More precisely, the main essential assumption we have used in the characterizations of Theorem 1 can be equivalently expressed, as we have already indicated in part I, by the condition $q(x) \leqslant \gamma |x|$ for all $x \in A$, γ constant. Now we shall see that even the mere finiteness of the function q is (necessary and) sufficient for the same conclusion. This is, in particular, a (formal) weakening of the assumption of uniform continuity of the spectral radius we have had in mind in part I; actually all these conditions are equivalent. But the present form 1^o is suitable for single elements.

Likewise condition 2^o we have already met before - even in its sharper form - but just now it appears in a natural relation to the commutativity theorems. Formally, assertion 2^o is an immediate consequence of 1^o and of Lemma 1 from part II.

Let us therefore outline the proof of 1^o; at the same time, this will provide another approach to the results of part I. From the first expression of q it is easily seen that q is a semi-norm on A (with possibly infinite values) such that $q(gxg^{-1}) = q(x)$ for g regular, x arbitrary, and $|x|_\sigma \leqslant q(x)$ for all $x \in A$.

If now $c \in Z(A)$, then $|c+y|_\sigma \leqslant |c|_\sigma + |y|_\sigma$ for all $y \in A$ (consider the corresponding classes in $A/\text{rad } A$) so that $q(c) = |c|_\sigma$ is finite.

Conversely, take an element $b \in A$ such that $q(b) < \infty$ and let us prove that $b \in Z(A)$. Let $a \in A$ be an arbitrary fixed element with no restriction on $q(a)$. Consider the entire function

$$f(\lambda) = e^{\lambda a} b e^{-\lambda a} = b + \lambda \delta_a(b) + \frac{\lambda^2}{2!} \delta_a^2(b) + \dots$$

where

$$\delta_a(b) = ab - ba \, .$$

Putting

$$h(\lambda) = \frac{f(\lambda) - b}{\lambda} = \delta_a(b) + \frac{\lambda}{2!} \delta_a^2(b) + \dots ,$$

which is again an entire function in λ , we can write

$$f(\lambda) = b + \lambda h(\lambda)$$

for all λ . Since $q(f(\lambda)) = q(b)$ we have

$$|h(\lambda)|_\sigma \leqslant q(h(\lambda)) \leqslant \frac{2q(b)}{|\lambda|}$$

for all $\lambda \neq 0$. According to the subharmonicity theorem of E. Vesentini [6] the function $|h(\lambda)|_\sigma$ is subharmonic on the whole complex plane, and since $q(b) < \infty$ we obtain from the above estimate that $|h(\lambda)|_\sigma \to 0$ if $|\lambda| \to \infty$. Hence it follows $|h(\lambda)|_\sigma = 0$ for all λ . In particular, for $\lambda = 0$ this gives $|ab - ba|_\sigma = 0$. This conclusion being true for all $a \in A$ we infer, using the most important result of C. Le Page [3] (cf. Lemma 2 in part II) that $ab - ba \in \operatorname{rad} A$ for all $a \in A$ which means that $b \in Z(A)$ as was to be proved. The rest of Theorem 4 is now obvious.

The corresponding local version of Theorem 1 can be proved in the following form.

THEOREM 5. Let A be an arbitrary Banach algebra. Let $c \in A$ be a fixed element. Then the following four conditions are equivalent:

1° $|c + y|_\sigma \leqslant \alpha(|c|_\sigma + |y|_\sigma)$ for all $y \in A$, α constant;

2° $|c(1 + y)|_\sigma \leqslant \beta \, |c|_\sigma |1 + y|_\sigma$ for all $y \in A$, β constant;

3° there is a constant γ such that $\big| |c + y|_\sigma - |y|_\sigma \big| \leqslant \gamma$ for all $y \in A$;

4° $c \in Z(A)$.

Again, it is enough to require these conditions not for all $y \in A$, but only for $y \in S(c)$.

To sum up: we have seen that the radical of a Banach algebra is characterized as the set of exactly those elements perturbations by which leave the spectrum completely invariant while the centre, a larger set, consists of just those elements perturbations by which can give rise to merely bounded changes of the spectrum. In other words, every element outside the centre can cause arbitrarily large changes of the spectrum; in particular, this refers to any element from $N \setminus \text{rad}\, A$. Moreover, from Theorem 2 we obtain immediately

COROLLARY 2. If $N \neq \text{rad}\, A$, then sums as well as products of two quasi-nilpotent elements can have arbitrarily large spectral radii.

The concepts of the radical and the centre arise in the general theory of abstract rings without topology where the notion of spectrum does not make a good sense. It is therefore all the more surprising that in the environment of Banach algebras these two general concepts, the centre and the radical, admit characterizations even in terms of the spectral radius, a phenomenon that can serve as one more evidence to the full recognition of the central role which the notion of spectrum or even the spectral radius itself play in the theory of Banach algebras. In the light of the results presented in this lecture, let us compare this role of the spectral radius in Banach algebra to that which plays, say, the force of gravity in Newton mechanics.

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ADDED IN PROOF (March 1977). Recently V. Müller has constructed a Banach algebra with $N = \text{rad } A = 0$ in which the spectral radius is discontinuous, answering thus negatively the question mentioned on p. 4 of this lecture.

Important applications of Theorem 4 will appear in the author's paper "Spectral characterization of two-sided ideals in Banach algebras" being submitted to Studia Mathematica. In that article we have found spectral characterizations for a number of other two-sided ideals similar to those obtained here for the radical and improving considerably those known as so called converse Weyl's theorems for the ideal of compact operators on a Banach space. The principal result can be formulated as follows.

THEOREM 6. Let I be a closed two-sided ideal of a Banach algebra A. Then the set of all elements $r \in A$ satisfying the condition

$$|a + I|_\sigma \leqslant |a + r|_\sigma \quad \text{for all} \quad a \in A$$

coincides with $\ker(\text{hul}(I))$, the intersection of all primitive ideals containing I.

In particular, it is interesting that, for a given $r \in A$, the inequality in Theorem 6 is equivalent with the (formally stronger) set inclusion

$$\sigma(a + I) \subset \sigma(a + r) \quad \text{for all} \quad a \in A .$$

Also, we believe this result may be helpful in connection with the characterization of elements belonging to two-sided ideals suggested on p. 6 of this lecture.

Further development of the present ideas will appear also in the great synthesis "Propriétés spectrales des algèbres de Banach" by Bernard Aupetit.