M. Atsuji

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ON NORMALITY OF THE PRODUCT OF TWO SPACES

M. ATSUJI

Tokyo

Let X be a T_1 space, Y a topological space, and p an accumulation point in X. We define an accumulation degree of p by

$$\alpha(p) = \min \{|A|;\ p \in \overline{A - \{p\}}\},$$

where $|A|$ means the cardianl of the set A.

$\omega(m)$ is the initial ordinal of an infinite cardinal m, and we define a property B*(m) of Y: For any decreasing closed family $\{F_\beta \subset Y;\ \beta \in \omega(m)\}$ with $\bigcap F_\beta = \phi$, there exists an open family $\{G_\beta \subset Y;\ \beta \in \omega(m)\}$ such that $F_\beta \subset G_\beta$ for each $\beta \in \omega(m)$ and $\bigcap \overline{G_\beta} = \phi$. The property B*($\aleph_0$) is nothing but the countable paracompactness([2]). A similar notion is defined by Zenor[5]: A space is said to have property B if for any well-ordered monotone decreasing family $\{H_\alpha;\ \alpha \in A\}$ of closed sets with no common part, there is a monotone decreasing family of domains $\{D_\alpha;\ \alpha \in A\}$ such that $H_\alpha \subset D_\alpha$ for each α in A and $\{cl(D_\alpha);\ \alpha \in A\}$ has no common part. (Yasui[4] defines a weak B-property which is obtained by removing the monotone decreasingness of the family of domains in Zenor's property B, and shows that Yasui's property is strictly weaker than Zenor's one). Zenor shows in [5] that the paracompactness implies the property B. Since the property B follows the property B*(m) for any m, a paracompact space has the property B*(m) for every m.

Let $\{E_x \subset Y;\ x \in Z \subset X\}$ be any family of subsets of Y with index set in X, and let p a point of X, then we write

$$\lim_p \sup E_x = \bigcap_{U \in N(p)} \overline{\bigcup_{x \in U} E_x},$$

where N(p) is the neighborhood system of p. Now we have

Proposition. Suppose $X \times Y$ is normal. Then for any closed family $\{E_x \subset Y;\ x \in Z \subset X\}$ with $\lim_p \sup E_x = \phi$ for a point p of X, there exists an open family $\{D_x \subset Y;\ x \in Z' \subset X\}$ satisfying $Z \subset Z'$, $E_x \subset D_x$ for $x \in Z$, and $\lim_p \sup \overline{D_x} = \phi$.

Proof. Let us put

$$E = \bigcup_{x \in Z} (x, E_x),$$

then $\overline{E}$ and (p, Y) are disjoint closed subsets of $X \times Y$ because

$$\overline{E}[p] = \lim_p \sup E_x = \phi \quad \text{(cf. [1])},$$

where $\overline{E}[p]$ is the slice of $\overline{E}$ at p. Therefore there exist disjoint open sets G_1 and G_2 containing $\overline{E}$ and (p, Y) respectively. Putting $D_x = G_1[x]$ for each $x \in pr_X G_1 = Z'$, we get the desired family $\{D_x;\ x \in Z'\}$. In fact, for an arbitrary point y of Y, there exist neighborhoods U and V of p and y respectively such that $U \times V \subset G_2$. V and D_x are disjoint for every x in U, so that $y \notin \overline{\bigcup_{x\in U} D_x} = \overline{\bigcup_{x\in U} \overline{D_x}}$; since y is arbitrary, we have

$$\bigcap_{U\in N(p)} \overline{\bigcup_{x\in U} \overline{D_x}} = \phi.$$

<u>Corollary 1</u>. Suppose X contains an accumulation point p with $\alpha(p) = m$. If $X \times Y$ is normal, then Y has the property B*(m).

<u>Proof</u>. p is an accumulation point of a subset A of X with $|A| = m$; we may assume p does not belong to A. Let $\{x_\beta;\ \beta\epsilon\omega(m)\}$ be a well-ordering of A, and let $\{F_\beta \subset Y;\ \beta\epsilon\omega(m)\}$ any decreasing closed family with no common part. Let us write $F_{x_\beta} = F_\beta$, then we have $\lim_p \sup F_{x_\beta} = \phi$. In fact,

$$\bigcap_{U\in N(p)} \overline{\bigcup_{x_\beta \in U} F_{x_\beta}} = \bigcap_{U\in N(p)} F_{x_{\beta_U}} = \bigcap_{\beta\epsilon\omega(m)} F_\beta = \phi,$$

where β_U is the least index of x_β belonging to U; the second equality is verified as follows: $q \notin \bigcap_{\beta\epsilon\omega(m)} F_\beta$ implies the existence of F_{β_0} which does not include q, and so $q \notin F_\gamma$ for all $\gamma \geq \beta_0$. Since $|\{x_\gamma;\ \gamma<\beta_0\}| < m$, p does not belong to $\overline{\{x_\gamma;\ \gamma<\beta_0\}}$, namely $\{x_\gamma;\ \gamma<\beta_0\}$ is disjoint from some neighborhood U_0 of p, so that $\beta_{U_0} \geq \beta_0$, and q is not in $F_{x_{\beta_{U_0}}}$ and not in $\bigcap_{U\in N(p)} F_{x_{\beta_U}}$.

Therefore, by Proposition, there exists an open family $\{G_x \subset Y;\ x\epsilon Z' \subset X\}$ such that $A \subset Z'$, $F_{x_\beta} \subset G_{x_\beta}$ for $x_\beta \epsilon A$, and $\lim_p \sup \overline{G_x} = \phi$. Since p is an accumulation point of A, we have

$$\bigcap_{x_\beta \epsilon A} \overline{G_{x_\beta}} = \phi;$$

in fact, suppose $\bigcap_{x_\beta \varepsilon A} \overline{G_{x_\beta}}$ includes a point y, then, since an arbitrary neighborhood U of p includes some x_β, y belongs to $\bigcup_{x \varepsilon U} \overline{G_x}$ and to $\bigcap_{U \varepsilon N(p)} \bigcup_{x \varepsilon U} \overline{G_x} = \phi$, the contradiction.

Corollary 2 (Dowker). Suppose X includes a sequence of points with an accumulation point in X. If X x Y is normal, then Y is countably paracompact.

About a month after this Prague Symposium the author received a pre-print[3] from Prof. M.Rudin in which the definition and the existence of a κ-Dowker space were given for an infinite cardinal κ. A κ-Dowker space is a normal T_2 space which has a decreasing closed family $\{D_\lambda;\ \lambda<\kappa\}$ such that $\bigcap D_\lambda = \phi$ and, if $\{U_\lambda;\ \lambda<\kappa\}$ is an open family with $D_\lambda \subset U_\lambda$ for each λ, then $\bigcap U_\lambda \neq \phi$. Since a κ-Dowker space of $\kappa = \aleph_{\omega(m)}$ does not have the property B*(m), we can immediately conclude by Corollary 1 above that the following Morita's conjecture is true.

Corollary 3 (Atsuji-Rudin). If X x Y is normal for any normal space Y, then X is a discrete space.

References

[1] M.Atsuji, Necessary and sufficient conditions for the normality of the product of two spaces, Proc. Japan Acad. 45(1969), 894-898.

[2] F.Ishikawa, On countably paracompact spaces, Proc. Japan Acad. 31(1955), 686-687.

[3] M.E.Rudin, κ-Dowker spaces, (pre-print).

[4] Y.Yasui, On the gaps between the refinements of the increasing open coverings, Proc. Japan Acad. 48(1972), 86-90.

[5] P.Zenor, A class of countably paracompact spaces, Proc. Amer. Math. Soc. 24(1970), 258-262.

JOSAI UNIVERSITY
SAKADO, SAITAMA
JAPAN