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Ostrava

If A is an integral domain with the quotient field K, the group of divisibility $G(A)$ of A is the partially ordered group $K^*/U(A)$, where K^* denotes the multiplicative group of K and $U(A)$ the group of units of A, with $aU(A) \leq bU(A)$ if and only if a divides b in A. It is well known that any abelian lattice-ordered group is a group of divisibility of some Bezout domain.

But, every lattice-ordered group may be endowed with the discrete topology and therefore considered a topological lattice-ordered group. We have an analogous situation for fields: every field may be considered a topological field with respect to the discrete topology.

Hence, it seems natural to consider the following question: does there exist for any abelian topological lattice-ordered group G a topological field $(K,\mathcal{T})$ and a Bezout domain A in K such that $U(A)$ is closed in K^* with respect to the topology induced from K and such that the factor group $K^*/U(A)$ is a topological lattice-ordered group isomorphic (i.e. group and lattice homeomorphic) with G ? In this case we say that G has a *representation* $(K,\mathcal{T},A)$.

We have solved this problem for a special class of topological lattice-ordered groups and special types of representations $(K,\mathcal{T},A)$ but, unfortunately, in general case the problem remains unsolved.

By a *topological lattice-ordered group* (notation: tl-group) we shall mean a triple $(G,\leq,\mathcal{F})$, where G is an abelian group, $(G,\leq)$ is a lattice ordered group (notation: l-group) and $\mathcal{F}$ is a topology on the underlying set $|G|$ of G such that $(G,\mathcal{F})$ is a topological group and $(|G|,\leq,\mathcal{F})$ is a topological lattice.

Two tl-groups are *tl-isomorphic* if there is a homeomorphism between them, which is both a lattice and group isomorphism.

If G is an l-group, then a *prime l-ideal* of G is a convex subgroup P of G, which is also a sublattice and from $\inf(a,b) \in P$ it follows $a \in P$ or $b \in P$ for any $a,b \in G$. Then a set $\{P_i : i \in J\}$ of prime l-ideals of a tl-group G is called a *topological realization* of G if P_i is closed in G for every $i \in J$, $\bigcap\{P_i : i \in J\} = \{0\}$ and the natural map

$$\pi : G \longrightarrow \Pi\{G/P_i : i \in J\}$$

is a tl-isomorphism from G onto πG, where πG inherits its topology, operation and ordering from $\Pi\{G/P_i : i \in J\}$.

Further, for any field K and a valuation w on K with value group G_w we may construct a field topology T_w in K defining the sets $U_{w,\alpha} = \{x \in K : w(x) > \alpha\}$, $\alpha \in G_w^+$, $R_w = \{x \in K : w(x) \geq 0\}$, as a base of the neighbourhoods of zero in K. Then the group $U(R_w)$ of units of R_w is open in K^* and $w : K^* \longrightarrow G_w$ is continuous with respect to the discrete topology on G_w.

First of all there holds the following proposition.

Proposition 1. *Let* H *be a closed l-ideal of a tl-group* G. *If* G *has a representation, then the factor tl-group* G/H *has a representation.*

P r o o f. If (K,T,A) is a representation of G, then by [2], Theorem 2.1. there exists a saturated multiplicative system S in A such that the group of divisibility of a quotient domain A_S is l-isomorphic with G/H. Then it is possible to show that (K,T,A_S) is a representation of G/H.

The following theorem solves completely the problem of existence of a representation for a totally ordered tl-group.

Theorem 2. *Let* G *be a totally ordered tl-group. Then* G *has a representation if and only if* G *is a discrete space.*

P r o o f. If (K,T,A) is a representation of G, then a canonical map $w : K^* \longrightarrow G$ is a continuous valuation. Since every set $\{\beta \in G : \beta > \alpha\}$, $\alpha \in G^+$, is open in G, it follows that $T_w \leq T$. Thus the set $U(R_w)$ is open in T and G is a discrete space.

It is well known that the factor group of an l-group with respect to a prime l-ideal is totally ordered. Thus, using Proposition 1 and Theorem 2 we obtain the following proposition.

Proposition 3. *If a tl-group* G *has a representation, then every closed prime l-ideal of* G *is open.*

Observe that using Proposition 3 an example of a tl-group (non-totally ordered), which has no representation, is easy to construct. The example of a topological product of two copies of a totally ordered group with the interval non-discrete topology works.

Now we shall find a condition for a tl-group G to have some special types of representations. We shall start with the following assertion.

Proposition 4. *Let* G *be a H-group and let* (K,T,A) *be its representation. Then there exists a topological realization* $\{P_i : i \in J\}$ *of* G *if and only if there is a family* $\{w_i : i \in J\}$ *of valuations of* K *such that* $\bigcap\{R_{w_i} : i \in J\} = A$, $T \geq \sup\{T_{w_i} : i \in J\}$ *and* $\{U(R_{w_i}) : i \in J\}$ *is a subbase for the sets* $U.U(A)$, *where* U *is an open neighbourhood of* 1_K *in* K^*.

We say that a representation (K,T,A) of a tl-group G is *locally bounded* provided that (K,T) is a locally bounded topological field and U(A) is a bounded set. Then the following theorem holds.

Theorem 5. *Let* G *be a tl-group with a topological realization* $\{P_i : i \in J\}$. *Then there exists a locally bounded representation of* G *if and only if* J *is a finite set and* P_i *is open for every* $i \in J$.

The p r o o f of this theorem is based on the using of Proposition 4.

Further, we say that a representation (K,T,A) is *locally compact* provided that (K,T) is a locally compact topological field. Then we have the following theorem.

Theorem 6. *Let* G *be a tl-group with a topological realization* $\{P_i : i \in J\}$. *Then there exists a locally compact representation of* G *if and only if* G *is a discrete tl-group isomorphic with the group* Z *of integers.*

P r o o f. If (K,T,A) is a locally compact representation of G, then by Proposition 4 there exists a family $\{w_i : i \in J\}$ of valuations such that $A = \bigcap\{R_{w_i} : i \in J\}$ and $\sup\{T_{w_i} : i \in J\} \leq T$. Every locally compact field is a complete topological field and it follows that T is a minimal field topology on K. Hence, $T \geq T_{w_i}$ for every $i \in J$ and the valuations w_i, $i \in J$, are mutually dependent. On the other hand, since T_{w_i} is locally compact, it follows that w_i is a discrete rank one valuation for every $i \in J$ and the valuations $w_i \neq w_j$ are independent. Thus, card J = 1 and $P_1 = \{0\}$. Therefore, $G \cong G/P_1 \cong G_{w_1} \cong Z$. The converse is evident.

It should be noted that we are not able to solve the problem of the existence of representation of a tl-group G even for special types of topologies on G. On the other hand, from the existence of such representation (K,T,A) it is possible to show some facts about the relations between the topology on G and a domain A.

We consider, for example, the topology $T_{\Pi'}$ on an l-group G such that the set Π' of all dual principal polars of G is a base of neighbourhoods of zero. Recall that a dual principal polar of G is a set $\{a\}' = \{g \in G : \inf(|g|,|a|) = 0\}$, where $a \in G$ and $|g| = \sup(g,-g)$.

Then the following proposition holds.

<u>Proposition 7</u>. *If a tl-group* $(G,T_{\Pi'})$ *has a representation* (K,T,A), *then* G *is a discrete space if and only if the Jacobson radical of* A *is non-zero.*

We conclude this note by mentioning a result of a continuous order relation in a topological group. Recall that for a partially ordered set $(M,\leq)$ with a topology T the order relation $\leq$ is called *continuous* if for any $a,b \in M$ such that $a \not\leq b$ there are $U,V \in T$ with $a \in U$, $b \in V$ such that for every $u \in U$, $v \in V$, $u \not\leq v$ holds. The importance of this notion follows from the fact that for every tl-group with T_2--topology the order relation in G is continuous.

We have the following simple characterization of continuous order relation in a topological order group.

<u>Proposition 8.</u> *Let* $(G,\leq,T)$ *be a topological order group. Then* $\leq$ *is continuous if and only if* G^+ *is closed in* G.

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