DESCRIPTIVE COMPLEXITY OF FUNCTIONS

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If continuous functions are considered as "simple", i.e. of minimal complexity, then the complexity of a discontinuous function f may be conceived as the minimal complexity of a suitable procedure by means of which f is obtained starting from continuous functions.

A well known procedure (the classical one of the descriptive theory) consists in transitions from sequences to their pointwise limits, every transition increasing the complexity by 1. Another procedure (see e.g.[1]) yields discontinuous functions as limits of filtered families of continuous ones. In this case, the complexity of f is, by definition, the least type of a filter yielding f. This kind of complexity is considered in Section 1. A broader concept also introduced in Section 1 is specialized to a certain kind of "quantitative" complexity in Section 2. It turns out that this kind of complexity includes e.g. the ε-entropy.

1.

1.1. With slight deviations, we use the standard terminology and notation. The ordered set of non-negative reals is denoted by R_+. The Fréchet filter on N, the set of natural numbers, is denoted by $\mathcal{N}$. If T is a set or a topological space or a metric space, then F(T) denotes the set of all real-valued functions on T endowed, as a rule, with the weak topology (the topology of R^T); M(T,T) denotes the set of all mappings $f: T \longrightarrow T$ endowed, in the case of a bounded metric T, with the "sup-distance" dist $(f,g) = \sup\{\text{dist}(f(t), g(t)): t \in T\}$; C(T) denotes the set of continuous $f \in F(T)$ endowed, as the case may be, either with the topology inherited from F(T) or with the "sup-distance".

We recall the following definitions (see e.g.[1]). If $\mathcal{F}$ and $\mathcal{G}$ are filters on A and B, respectively, then a morphism from $\mathcal{F}$ to $\mathcal{G}$ is a triple $\langle \varphi, \mathcal{F}, \mathcal{G} \rangle$ denoted often by $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ or simply by φ , such that φ is a mapping of A into B and $\varphi^{-1}[G] \in \mathcal{F}$ whenever

$G \in \mathcal{G}$. The types of filters are introduced as follows: Typ $\mathcal{F} \geqq$ $\geqq$ Typ $\mathcal{G}$ iff there is a morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$, and Typ $\mathcal{F}$ = Typ $\mathcal{G}$ iff Typ $\mathcal{F} \geqq$ Typ $\mathcal{G} \geqq$ Typ $\mathcal{F}$. In this way, the class of all types of filters is endowed with an order.

1.2. Proposition. In the ordered class of all filter-types, the join (meet) exists for every (every non-void) subset. - This follows at once e.g. from [2], 2.2.

1.3. Definition. If $V = \langle V, \prec \rangle$ is an ordered set, then a mapping $\mu : A \longrightarrow V$, where A is a set, will be called a V-evaluation on A and $\langle A, \mu \rangle$ will be called a V-evaluated set. If $\mathcal{F}$ and μ are, respectively, a filter and a V-evaluation on a set A, then $\langle \mathcal{F}, A, \mu \rangle$ will be called a V-evaluated filter. If $\mathbb{F} = \langle \mathcal{F}, A, \mu \rangle$, $\mathbb{G} = \langle \mathcal{G}, B, \nu \rangle$ are V-evaluated filters, then a triple $\langle \varphi, \mathbb{F}, \mathbb{G} \rangle$ will be called a morphism from $\mathbb{F}$ to $\mathbb{G}$ provided (1) $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism, (2) $\nu(\varphi(a)) \prec \mu(a)$ for all $a \in A$. For a fixed V, the types of V-evaluated filters and the order on the class of these types are introduced by means of morphisms in the same way as for filters (see 1.1). Explicit definitions may be omitted.

1.4. Proposition. For a fixed V, the class of all types of V-evaluated filters is relatively complete, i.e. every subset has a unique join and every non-void subset has a unique meet.

This follows at once from the obvious "V-evaluated" generalization of [2], 2.2.

1.5. Definition. Let P be a closure space and let $S \subset P$ be dense in P. Let $x \in P$. Consider all filters $\mathcal{F}$ such that, for some family $(s_a : a \in A)$, $s_a \in S$, we have $x = \mathcal{F}\text{-lim } s_a$. The meet (the greatest lower bound) of types of such filters will be called the descriptive complexity of $x \in P$ with respect to S and will be denoted by dc (x,P,S). Let T be a functionally Hausdorff topological space. If $f \in F(T)$, then dc (f,F(T),C(T)) will be called the descriptive complexity of f (as a function on T) and will be denoted by dc f.

1.6. Remarks. 1) It is sometimes useful to introduce a related concept obtained by considering, in 1.5, only filters $\mathcal{F}$ of a certain class (e.g. filters on countable sets). - 2) A variety of descriptive complexity is obtained if, in 1.5, "topological" is replaced by "uniform" and C(T) consists of uniformly continuous functions. - 3) If T is of uncountable weight, it may happen that dc f < Typ $\mathcal{N}$ for a function f of the first Baire class. - 4) Clearly, if f is of Baire class α, then dc f $\leqq \alpha$ = Typ $\mathcal{N}^{\alpha}$ (see [1], 5.1, 5.2). I do not know under what additional conditions dc f = α can be asserted.

1.7. Theorem. The descriptive complexity of a function f on T is equal to the type of the filter of all $U \cap C(T)$ where U is a neighborhood of f in $F(T)$.

1.8. Theorem. Let f_a, $a \in A$, be functions on a topological space T. Let φ be a continuous function on the topological space R^A. Put $g(t) = \varphi(f_a(t): a \in A)$ for $t \in T$. Then the descriptive complexity of g does not exceed the l.u.b. of dc (f_a).

1.9. Corollary. For any topological space T and $f, g \in F(T)$, dc $(f + g) \leqq \sup$ (dc f, dc g).

1.10. We recall that, for any filters $\mathcal{F}$ and $\mathcal{G}$ (on A and B, respectively), $\mathcal{F} \cdot \mathcal{G}$ designates the filter on $A \times B$ consisting of all $X \subset A \times B$ such that, for some $F \in \mathcal{F}$, $a \in F$ implies $\{b: \langle a,b \rangle \in X\} \in \mathcal{G}$. We put Typ $(\mathcal{F}.\mathcal{G})$ = Typ $\mathcal{F}$. Typ $\mathcal{G}$.

1.11. Theorem. Let f_a, $a \in A$, and f be functions on T. Let $\mathcal{F}$ be a filter on A; assume that $f = \mathcal{F}\text{-}\lim f_a$. Then dc $(f) \leqq$ Typ $\mathcal{F}$. sup dc (f_a).

The proofs of theorems 1.7, 1.8, 1.11 are straightforward and therefore omitted.

1.12. Theorem. If T is both F_σ and G_δ in a compact metrizable space, then the descriptive complexity of any function on T is equal to the type of a filter on a countable set.

We are going to prove that the assertion holds if T is assumed to be compact metrizable. From this special case (stated without proof in [1], 4.8), the theorem follows easily. Now let $f \in F(T)$, T compact metrizable, f non$\in C(T)$. Choose a countable M dense in $C(T)$ endowed with the usual sup-norm $|x| = \sup |x(t)|$. Let $\mathcal{W}(f)$ be the filter of all neighborhoods of f in $F(T)$. Clearly, Typ $\{W \cap M: W \in \mathcal{W}(f)\} \geqq$ Typ $\mathcal{W}(f)$. We are going to prove that there is a morphism of $\mathcal{W}(f)$ into $\mathcal{W}(f) \restriction M = \{W \cap M: W \in \mathcal{W}(f)\}$.

Choose distinct $a_k \in T$, $k \in N$, such that f restricted to $\{a_k\}$ is not continuous. For any $g \in C(T)$ let $p(g)$ be the largest $n \in N$ such that $|g(a_k) - f(a_k)| \leqq (n+1)^{-1}$ for $k < n$ ($p(g)$ is defined correctly since f cannot coincide with the continuous function g on all a_k).

Now let $\varphi : C(T) \longrightarrow M$ be such that, for any $g \in C(T)$, we have $\varphi(g) \in M$, $|g - \varphi(g)| \leqq 1/p(g)$. We are going to show that φ is a morphism from $\mathcal{W}(f)$ to $\mathcal{W}(f) \restriction M$. It is sufficient to show that, given $\tau \in T$ and $\varepsilon > 0$, there is a finite $K \subset T$ and a positive δ such that $|\varphi(g)(\tau) - f(\tau)| < \varepsilon$ whenever $|g(t) - f(t)| < \delta$ for all $t \in K$. It is easy to see that $K = \{\tau, a_0, \dots, a_q\}$, where $q > 2/\varepsilon$, and $\delta = \frac{1}{2}$ satisfy the condition just mentioned.

Remark. I do not know what conditions on T broader than those in 1.12 are sufficient to ensure that all dc f, $f \in F(T)$, are types of filters on countable sets.

1.13. We introduce the following pre-order on the class $\bigcup F(T)$ of all real-valued functions on topological spaces: if $f, g \in \bigcup F(T)$, we put $f \preccurlyeq g$ iff there exists a continuous $h: Df \longrightarrow Dg$ (Df, Dg are domains of f, g) such that $f = g \circ h$.

1.14. Theorem. For every filter-type ξ there exists a function λ such that dc $f \leqq \xi$ if and only if $f \preccurlyeq \lambda$.

Proof. Let $\xi = \mathrm{Typ}\,\mathcal{F}$, $\mathcal{F}$ being a filter on a set A. Let $S \subset R^A$ consist of all $(x_a: a \in A)$ such that $\mathcal{F}$-lim x_a exists (and is in R). For any $x = (x_a) \in S$, put $\lambda(x) = \mathcal{F}\text{-lim}\, x_a$. It is easy to show that, for any $f \in \bigcup F(T)$, dc $f \leqq \mathrm{Typ}\,\mathcal{F}$ iff $f \preccurlyeq \lambda$.

2.

In this section we introduce a rather special kind of complexity which is different from the descriptive one (see 1.5) and could be called "quantitative". The theorems below, though almost evident (once the definitions are stated), show that this concept embraces some important cases.

2.1. Put $V = R_+ \times R_+$. Consider the class, denoted by Φ, of all $\langle \mathcal{N}, N, \varphi \rangle$, where φ is a V-evaluation on N and, with $\varphi(k) = \langle \mu(k), \lambda(k) \rangle$, we have $\lambda(k) \longrightarrow 0$ for $k \longrightarrow \infty$. If $\langle \mathcal{N}, N, \varphi \rangle \in \Phi$, then ψ_φ will designate the function on R_+ defined by $\psi_\varphi(\varepsilon) = \inf(\mu(k): k \in N, \lambda(k) \leqq \varepsilon)$; for $\varepsilon = 0$, the value $\psi_\varphi(0) = \infty = \inf \emptyset$ is admitted, whereas for $\varepsilon > 0$, we have $\psi_\varphi(\varepsilon) \in R_+$ (since $\lambda(k) \longrightarrow 0$ for $k \longrightarrow \infty$). Clearly, $\psi_\varphi(\varepsilon_1) \geqq \psi_\varphi(\varepsilon_2)$ if $\varepsilon_1 \leqq \varepsilon_2$.

2.2. Proposition. If $\langle \mathcal{N}, N, \varphi_i \rangle \in \Phi$, $i = 1,2$, then $\mathrm{Typ}\,\langle \mathcal{N}, N, \varphi_1 \rangle \geqq \mathrm{Typ}\,\langle \mathcal{N}, N, \varphi_2 \rangle$ (see 1.3) if and only if $\psi_{\varphi_1} \geqq \psi_{\varphi_2}$.

The proof is easy and may be omitted.

2.3. By 2.2, we may adopt the following convention: if $\langle \mathcal{N}, N, \varphi \rangle \in \Phi$, we put $\mathrm{Typ}\,\langle \mathcal{N}, N, \varphi \rangle = \psi_\varphi$.

2.4. Definition. Let P be a set, $S \subset P$. Let μ be an R_+-evaluation on S and let σ be an R_+-evaluation on a set $Q \supset P \times S$. Let $x \in P$ and assume that the set Σ_x of all sequences (s_n), $s_n \in S$, such that $\sigma(x, s_n) \longrightarrow 0$ is non-void. For any $\sigma = (s_n) \in \Sigma_x$ define a V-evaluation φ_σ on N as follows: $\varphi_\sigma(n) = \langle \mu(s_n), \sigma(x, s_n) \rangle$. Clearly, $\langle \mathcal{N}, N, \varphi_\sigma \rangle \in \Phi$. The g.l.b. of all $\mathrm{Typ}\,\langle \mathcal{N}, N, \varphi_\sigma \rangle$, $\sigma \in \Sigma_x$, will be called the complexity of x with respect to P, S, μ.

(complexity of elements of S), and σ . (If some of P, S, μ, σ are clear from the context, then they need not be mentioned explicitly.) - If the complexity, say ξ , of x is equal to some Typ $\langle \mathcal{N}, N, \wp \rangle$, then, by 2.3, $\xi = \psi_{\wp}$, and hence $\lim \xi = \lim \psi_{\wp}(\varepsilon)$ for $\varepsilon \longrightarrow 0$ is defined. It will be called the limit complexity of x.

Remark. Observe that the complexity of x may be distinct from all types Typ $\langle \mathcal{N}, N, \wp \rangle$. However, in all cases considered below, there is a smallest element in the set of all types Typ $\langle \mathcal{N}, N, \wp_{\sigma} \rangle$, $\sigma \in \Sigma_x$.

2.5. We are now going to consider ε-entropy (2.6, 2.7), metric dimension (2.8, 2.9) and approximation of continuous functions by polynomials of prescribed degree (2.10, 2.11).

Concerning ε-entropy, introduced by A.N. Kolmogorov in 1956, and degrees of polynomial approximation $E_n(f)$, considered in detail for the first time independently by S.N. Bernstein and D. Jackson about 1912, basic facts can now be found in various books, see e.g. [3]. For metric dimension see e.g. [4].

2.6. There are various slightly different definitions of metric entropy (ε-entropy) of a totally bounded metric space T. We will use the following one: for every $\varepsilon \geqq 0$, the ε-entropy $H(\varepsilon, T)$ of T is equal to $\log N(\varepsilon, T)$ where $N(\varepsilon, T)$ is the least cardinality of an ε-net in T (thus, $H(0,T) = \infty$ provided T is infinite), the metric entropy of T is the function $\varepsilon \longrightarrow H(\varepsilon, T)$ defined on R_+.

2.7. Theorem. Let T be a totally bounded non-void metric space. The metric entropy of T is equal to the complexity of the identity mapping J_T: $T \longrightarrow T$ with respect to the set of all finite-range mappings g: $T \longrightarrow T$, their complexity defined as log card $g[T]$, and the sup-distance.

This theorem is proved in a straightforward way using only the definitions and almost no facts concerning the ε-entropy. For this reason, the proof of 2.7, as well as of 2.9 and 2.10 below is omitted.

2.8. We recall that the metric dimension $\mu \dim T$ of a totally bounded metric space T is defined as follows: $\mu \dim T \leqq m$ iff, for every $\varepsilon > 0$, there exists a finite open covering $\mathcal{G}$ of T such that (1) diam $G \leqq \varepsilon$ for all $G \in \mathcal{G}$, (2) $\mathcal{G}$ is of order $\leqq m + 1$.

2.9. Theorem. The metric dimension of a totally bounded metric space T is equal to the limit complexity of the identity mapping J: $T \longrightarrow T$ with respect to finite-range mappings g: $T \longrightarrow T$, their complexity defined as the order of $\{ \mathrm{c}\ell\, g^{-1}(t) : t \in g[T] \}$ minus 1, and the sup-distance (in M(T,T)).

2.10. We recall that if $K \subset R$ is compact non-void, then for any $f \in C(K)$ and any $n \in N$, $E_n(f)$ designates the g.l.b. of $\| f - p \|$, where p is a polynomial of degree $\leqq n$.

2.11. Theorem. Let $K \subset R$ be compact non-void. Then, for any $f \in C(K)$ and any n, $E_n(f) = \inf \{ \varepsilon : \psi(\varepsilon) \leqq n \}$ where ψ is the complexity of f with respect to the set of polynomials, their degree, and the sup-distance in C(K).

References

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