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Equicontinuity and the Theorem of ARZELA-ASCOLI in uniform convergence spaces

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In this paper I want to give a general formulation of the classical Theorem of ARZELA-ASCOLI in the context of convergence spaces. For the proofs the reader will be refered to my paper [7] and for details on convergence spaces to FISCHER [5] and W. GÄHLER [6] . By the classical Theorem of ARZELA-ASCOLI, a subset H of the space of the continuous mappings of a closed interval into the reals or the complex numbers is relatively compact with respect to the uniform convergence if and only if

(A) H is uniformly bounded

and

(B) H is equicontinuous.

In the known generalizations as well as in the following one, H is a subset of a general function space C(X,Y), the uniform convergence in C(X,Y) is substituted by the continuous convergence, and (A) is substituted by

(A') H(x) is relatively compact for all x of X.

COOK and FISCHER [4] proved the Theorem for the case in which X is a pseudo-topological space and Y is a HAUSDORFF uniform space in the sense of BOURBAKI. SIMONNET [10] proved the Theorem for the case in which X is a pseudo-topological space and Y is a pseudo-topological linear space with the CHOQUET condition. POPPE [9] gave a generalization for generalized uniform spaces in the sense of TUKEY and MORITA.

1. In the following, the notion of pseudo-topology is used in the sense of FISCHER [5] and the notion of uniform convergence structure is used in the sense of COOK and FISCHER [4]. For any set X, the filter in X consisting of all subsets of X containing a fixed set M is denoted by [M], the diagonal in $X \times X$ is denoted by Δ_X. Pseudo-topologies (and uniform convergence structures, too) on the same set X will be set-theoretically ordered. A pseudo-topology σ on X is called finer than a pseudo-topology τ on X and τ is called coarser than σ if $\tau(x)$ contains $\sigma(x)$ for all $x \in X$ and a uniform convergence

structure $\breve{u}$ on X is called finer than a uniform convergence structure $\mathfrak{w}$ on X and $\mathfrak{w}$ is called coarser than $\breve{u}$ if $\mathfrak{w}$ contains $\breve{u}$. For any pseudo-topological space (X,τ), we denote the finest topology on X coarser than τ by $t(\tau)$. A uniform convergence space $(X,\breve{u})$ is called an uniform CHOQUET space if a filter $\mathfrak{U}$ in $X\times X$ belongs to $\breve{u}$ if and only if every ultra-filter $\mathfrak{V}\geq\mathfrak{U}$ belongs to $\breve{u}$. The pseudo-topology $\lambda(\breve{u})$ induced by a uniform CHOQUET structure $\breve{u}$ on X is a CHOQUET pseudo-topology, that means a pseudo-topology such that a filter $\mathfrak{F}$ in X converges to a point of X if and only if every ultra-filter containing $\mathfrak{F}$ converges to the same point. Pseudo-topological CHOQUET vector spaces are examples of uniform CHOQUET spaces. Let (X,τ) be a pseudo-topological vector space. Then the mapping $w:(x,y)\mapsto x-y$ defines a - canonical - uniform convergence structure $\breve{u}_\tau$ on X with $\mathfrak{U}\in\breve{u}_\tau$ if and only if $w(\mathfrak{U})\in\tau(0)$ (see W. GÄHLER [6], Bd. 2). $\breve{u}_\tau$ induces the vector pseudo-topology τ on X and $\breve{u}_\tau$ is a uniform CHOQUET structure if and only if τ is a CHOQUET pseudo-topology. This remark insures that the result of SIMONNET is a special case of our theorem. A uniform convergence space $(X,\breve{u})$ is called uniformly regular if with any filter $\mathfrak{U}\in\breve{u}$ the adherence $\overline{\mathfrak{U}}$ - relative to the pseudo-topology $\lambda(\breve{u})\times\lambda(\breve{u})$ - belongs to $\breve{u}$.

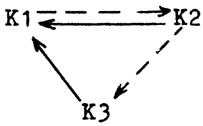
2. In the following, three natural notions of the relative compactness of a subset M in a pseudo-topological space (X,τ) appear.

(K1) Every ultra-filter $\mathfrak{F}$ in X with $M\in\mathfrak{F}$ converges (M is relatively compact in the generalized sense).

(K2) The adherence of M with respect to the pseudo-topology τ of X is compact (M is relatively compact).

(K3) The adherence of M with respect to the finest topology $t(\tau)$ coarser than τ is compact (M is t-relatively compact).

In the diagram



(K1) implies (K2) if X is regular (there are counter-examples even for topological spaces, see BOURBAKI [2]). (K2) implies (K3) if X is separated.

3. Let (X,σ) be a pseudo-topological space and $(Y,\check{\mathcal{U}})$ a uniform convergence space. For every $x\in X$ we define a mapping $\varphi_x : C(X,Y)\times X \to Y\times Y$ by $\varphi_x(f,x') = \big(f(x),f(x')\big)$. Then a subset H of C(X,Y) is called equicontinuous in the sense of COOK and FISCHER if for every $x\in X$ and every filter $\mathfrak{F}$ converging to x the filter $\varphi_x([H]\times\mathfrak{F})$ belongs to $\check{\mathcal{U}}$. Now we are able to formulate the

Theorem 1. Let (X,σ) and (Y,τ) be pseudo-topological spaces and let H be a subset of C(X,Y) (C(X,Y) is equipped with the continuous convergence).

(i) If Y is separated then H(x) is t-relatively compact for every $x\in X$ if H is t-relatively compact.

For the further assertions let $\check{\mathcal{U}}$ be an uniform convergence structure on Y and $\tau=\lambda(\check{\mathcal{U}})$ the induced pseudo-topology.

(ii) If $(Y,\check{\mathcal{U}})$ is a uniform CHOQUET space then H is equicontinuous if H is relatively compact in the generalized sense.

(iii) If $(Y,\check{\mathcal{U}})$ is uniformly regular, $(Y,\lambda(\check{\mathcal{U}}))$ is a CHOQUET space, and $(Y,t(\lambda(\check{\mathcal{U}})))$ is separated then H is relatively compact if H is equicontinuous and H(x) is t-relatively compact for all $x\in X$.

Proving the Theorem the following useful known property of CHOQUET spaces is used: Let (Y,τ) be a pseudo-topological CHOQUET space such that $t(\tau)$ is separated. Then τ and $t(\tau)$ agree on any compact subset of Y (see COOK [2]). We remark that every uniform CHOQUET space $(Y,\check{\mathcal{U}})$ has an analogous property (see [8]): If the finest uniform structure $\mathcal{W}$ on X coarser than $\check{\mathcal{U}}$ is separated then $\check{\mathcal{U}}$ and $\mathcal{W}$ agree on any compact subset of Y.

For any separated and regular space Y the space C(X,Y) is separated and regular, too. Finally, respecting the equivalence of the three notions of relative compactness for Y and C(X,Y) instead of X, we get the

Theorem 2. Let (X,σ) be a pseudo-topological space and let $(Y,\check{\mathcal{U}})$ be a uniformly regular uniform CHOQUET space such that $t(\lambda(\check{\mathcal{U}}))$ is separated. Then a subset H of C(X,Y) is relatively compact if and only if H is equicontinuous and H(x) is relatively compact for all x of X.

The theorem of SIMONNET for an arbitrary pseudo-topological space X and a regular CHOQUET vector space (Y,τ) is a special case of our theorem if we use the canonical uniform convergence structure $\check{\mathcal{U}}_\tau$. τ is regular if and only if $\check{\mathcal{U}}_\tau$ is uniformly regular such that our theorem can be applied.

4. Finally, following ANANTHARAMAN and NAIMPALLY [1], we give a useful characterization of the equicontinuity by means of the notion of nonexpansiveness (I am indebted to Prof. S. A. NAIMPALLY for the information about his results in uniform spaces).

Let G be a family of mappings of a set X into X. For any subset U of $X \times X$ we define

$$U_G = \bigcap_{g \in G} \{(x,y) \mid (g(x),g(y)) \in U\} \cap U .$$

For any filter $\mathfrak{U}$ in $X \times X$ let $\mathfrak{U}_G$ be the filter in $X \times X$ with the base $\{U_G \mid U \in \mathfrak{U}\}$. Then we have the

Lemma. Let $(X,\check{\mathcal{U}})$ be a uniform convergence space and G be a family of mappings of X into X. Then the system $\{\mathfrak{U}_G \mid \mathfrak{U} \in \check{\mathcal{U}},\ \mathfrak{U} \subseteq [\Delta_X]\}$ is a base of a uniform convergence structure $\check{\mathcal{U}}_G$ on X finer than $\check{\mathcal{U}}$.

Proof. For any subset U of $X \times X$ with $\Delta \subseteq U$, we have $\Delta \subseteq U_G$ and hence $[\Delta] \supseteq \mathfrak{U}_G$ for any filter $\mathfrak{U}$ in $X \times X$. For two subsets U and V of $X \times X$ we have $U_G \cup V_G \subseteq (U \cup V)_G$ and $U_G \circ V_G \subseteq (U \circ V)_G$ and hence $\mathfrak{U}_G \cap \mathfrak{V}_G \supseteq (\mathfrak{U} \cap \mathfrak{V})_G$ and $(\mathfrak{U}_G \circ \mathfrak{V}_G) \supseteq (\mathfrak{U} \circ \mathfrak{V})_G$, respectively, for two filters $\mathfrak{U}$ and $\mathfrak{V}$ in $X \times X$. Consequently, $\{\mathfrak{U}_G \mid \mathfrak{U} \in \check{\mathcal{U}},\ \mathfrak{U} \subseteq [\Delta_X]\}$ is the base of an uniform convergence structure on X. On account of $U_G \subseteq U$ for all subsets U of $X \times X$ we get $\mathfrak{U}_G \supseteq \mathfrak{U}$ for any filter and therefore $\check{\mathcal{U}}_G$ is finer than $\check{\mathcal{U}}$.

Remark. If we take $(g \times g)(x,y) = (g(x),g(y))$ for a mapping $g: X \to X$, obviously we have $(g \times g)(U_G) \subseteq U$ for all $g \in G$, hence $(g \times g)(\mathfrak{U}_G) \supseteq \mathfrak{U}$, and hence $(g \times g)(\check{\mathcal{U}}_G) \subseteq \check{\mathcal{U}}$. That means the uniform continuity of all the mappings $g \in G$ with respect to $\check{\mathcal{U}}_G$ and $\check{\mathcal{U}}$ and therefore $\check{\mathcal{U}}_G$ is finer than the uniform convergence structure initiated by the family G (see W. GÄHLER [6], Bd. 1).

Definition. A family G of mappings of a uniform convergence space X into X is said to be nonexpansive with respect to the uniform convergence structure $\check{\mathcal{U}}$ of X if there is a uniform convergence structure $\mathcal{W}$ on X with the following properties:

(N1) $\mathfrak{n}$ is finer than $\check{\mathfrak{u}}$.

(N2) $\check{\mathfrak{u}}$ and $\mathfrak{n}$ induce the same pseudo-topology $\lambda(\check{\mathfrak{u}})=\lambda(\mathfrak{n})$ on X.

(N3) There is a base $\mathfrak{m}$ of $\mathfrak{n}$ such that for every $\mathfrak{M} \in \mathfrak{m}$, $W \in \mathfrak{M}$, and $g \in G$

$$(x,y) \in W \Longrightarrow (g(x),g(y)) \in W.$$

__Theorem 3__. A semigroup G of mappings of a uniform convergence space X into X is equicontinuous if and only if it is nonexpansive.

__Proof.__ Let $\check{\mathfrak{u}}$ be the uniform convergence structure of X. First, let G be nonexpansive and let $\mathfrak{n}$ and $\mathfrak{m}$ be chosen according to the definition of nonexpansiveness. Let a filter $\mathfrak{F}$ in X be convergent to $x \in X$ with respect to $\check{\mathfrak{u}}$. Then $\mathfrak{F}$ converges to x with respect to $\mathfrak{n}$, too, and hence there is a filter $\mathfrak{M}$ of the base $\mathfrak{m}$ of $\mathfrak{n}$ with $[x] \times \mathfrak{F} \geqq \mathfrak{M}$. For every $W \in \mathfrak{M}$ there is an $F \in \mathfrak{F}$ with $\{x\} \times F \subseteq W$ and according to (N3) we have $\varphi_X(G \times F) \subseteq W$ and therefore $\varphi_X([G] \times \mathfrak{F}) \geqq \mathfrak{M}$. Because of (N1), this implies the equicontinuity of G (with respect to $\check{\mathfrak{u}}$).

On the other hand, let G be equicontinuous. Then the structure $\check{\mathfrak{u}}_G$ introduced in the Lemma is finer than $\check{\mathfrak{u}}$ and hence $\lambda(\check{\mathfrak{u}}_G)$ is finer than $\lambda(\check{\mathfrak{u}})$. Let a filter $\mathfrak{F}$ be convergent to x with respect to $\check{\mathfrak{u}}$. Because of $\{x\} \times F \subseteq ((\{x\} \times F) \cup \varphi_X(G \times F))_G$ for all $F \in \mathfrak{F}$, the filter $[x] \times \mathfrak{F}$ contains the filter $(([x] \times \mathfrak{F}) \cap \varphi_X([G] \times \mathfrak{F}) \cap [\Delta_X])_G$ belonging to $\check{\mathfrak{u}}_G$. Thus $\mathfrak{F}$ converges to x with respect to $\check{\mathfrak{u}}_G$ and $\lambda(\check{\mathfrak{u}})$ and $\lambda(\check{\mathfrak{u}}_G)$ agree. Finally, for $\mathfrak{U} \in \check{\mathfrak{u}}$ with $[\Delta_X] \geqq \mathfrak{U}$ and $U \in \mathfrak{U}$ we have $(u,v) \in U_G \Longrightarrow (g(u),g(v)) \in U \Longrightarrow (f(g(u)),f(g(v))) \in U$ for all $f,g \in G$ and hence $(u,v) \in U_G \Longrightarrow (g(u),g(v)) \in U_G$ for all $g \in G$. This proves G being nonexpansive (with respect to $\mathfrak{n} = \check{\mathfrak{u}}_G$ and the base $\mathfrak{m} = \{\mathfrak{U}_G \mid \mathfrak{U} \in \check{\mathfrak{u}}, [\Delta_X] \geqq \mathfrak{U}\}$.

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