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1. History; Bibliography

E-compact spaces were first defined in [7]. A related concept, that of E-complete regularity, was discussed in [8] without, however, formulating an explicit definition. The first systematic investigation of both concepts as well as explicit definitions were given in [6]. A partial summary of results on the topological aspects of the theory is given in [9]. Algebraic aspects are studied in [5]; set-theoretic (closely related to Ulam non measurability) in [11], [12], [13]. Certain generalizations of these concepts are discussed in papers of Herrlich [3], [4].

In the following sections we shall attempt to give a brief description of the basic results as well as to state unsolved problems and indicate probable directions of further development.

2. The Embedding Theorem

We are concerned with embedding into products. Maps into a product can be considered as collections of maps into coordinate spaces. Given a map h from a space X into a product $\times\{E_\xi\colon \xi \in \Xi\}$, we define

$$f_\xi = \pi_\xi \circ h \quad \text{for every} \quad \xi \in \Xi \tag{1}$$

where π_ξ denotes the projection of $\times\{E_\xi : \xi \in \Xi\}$ onto E_ξ (and $\pi_\xi \circ h$ denotes the composition of π_ξ with h). In such a way we obtain a collection $\mathfrak{F} = \{f_\xi\colon \xi \in \Xi\}$ of functions such that $f_\xi\colon X \to E_\xi$. Conversely, given such a collection $\mathfrak{F}$ of functions there exists exactly one map h into the product satisfying (1). For obvious reasons, we shall call h the parametric map corresponding to $\mathfrak{F}$. The purpose of the embedding theorem is to express some properties of the parametric map in terms of properties of the class $\mathfrak{F}$.

The Embedding Theorem. *Let $\mathfrak{F} = \{f_\xi\colon \xi \in \Xi\}$ be a class of functions with f_ξ: $X \to E_\xi$. Let h be the parametric map corresponding to the class $\mathfrak{F}$. We have*

(a) *h is continuous if, and only if, each* f_ξ *is continuous;*

(b) *h is one-to-one if, and only if, the class* $\mathfrak{F}$ *satisfies the following condition:*

(i) *for every* $p, q \in X$, $p \neq q$, *there is an* $f_\xi \in \mathfrak{F}$ *with* $f_\xi(p) \neq f_\xi(q)$;

(c) *h is a homeomorphism if, and only if, h is continuous and one-to-one and the class* $\mathfrak{F}$ *satifies the following condition:*

(ii) *for every closed subset A of X and for every* $p \in X - A$, *there exists a finite system* $f_{\xi_1}, \ldots, f_{\xi_n}$ *of functions from F such that* $\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle (p) \notin \operatorname{cl} \langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle [A]$,[1]) *where* cl *denotes the closure in* $E_{\xi_1} \times \ldots \times E_{\xi_n}$;

(d) *assume that the spaces* E_ξ *are all Hausdorff and assume that h is a homeomorphism;* $h[X]$ *is closed in* $\times\{E_\xi\colon \xi \in \Xi\}$ *if, and only if the class* $\mathfrak{F}$ *satisfies the following condition:*

(iii) *there is no proper extension* εX *of X such that every function* $f_\xi \in \mathfrak{F}$ *admits a continuous extension* $f_\xi^*\colon \varepsilon X \to E_\xi$.

Furthermore, in condition (iii), *it suffices to consider only extensions* εX *of X such that* $\varepsilon X \underset{\mathrm{top}}{\subset} \times\{E_\xi\colon \xi \in \Xi\}$.

In connection with the embedding theorem we shall introduce the following definition.

Definition. *An* $\{E_\xi\colon \xi \in \Xi\}$*-distinguishing, an* $\{E_\xi\colon \xi \in \Xi\}$*-separating, an* $\{E_\xi\colon \xi \in \Xi\}$*-non-extendable class for X is a class* $\mathfrak{F} = \{f_\xi : \xi \in \Xi\}$ *of continuous functions with* $f_\xi\colon X \to E_\xi$ *satisfying condition* (i), (ii), (iii) *of the Embedding Theorem, respectively. If all the spaces* E_ξ *are equal to a fixed space E, then we shall use the terms: an E-distinguishing, an E-separating, an E-non-extendable class.*

3. *E*-Completely Regular Spaces

Given two spaces X and E, we say that X is *E-completely regular* provided that X can be embedded into some power of E. The class of all E-completely regular spaces will be denoted by $\mathfrak{C}(E)$. Classes of spaces of the form $\mathfrak{C}(E)$ will be called classes of complete regularity.

The following are some examples of classes of complete regularity. The class of all completely regular spaces is a class of complete regularity; namely, it coincides with $\mathfrak{C}(I)$ where $I = [0, 1]$. The class of all 0-dimensional T_0-spaces is again a class of complete regularity, namely, $\mathfrak{C}(D)$, where D is the two-point discrete space. The class $\mathfrak{C}(F)$ is equal to the class of all T_0-spaces, where F is the two-point space in which

[1]) If $f_{\xi_1}, \ldots, f_{\xi_n}$ is a finite system of functions with $f_{\xi_i}\colon X \to E_{\xi_i}$, then by $\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle$ we shall mean a map whose value at a point $p \in X$, $\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle (p)$, is equal to the point $(f_{\xi_1}(p), \ldots, f_{\xi_n}(p))$ of the product $E_{\xi_1} \times \ldots \times E_{\xi_n}$ (i.e., $\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle$ is the parametric map corresponding to the class $\langle f_{\xi_1}, \ldots, f_{\xi_n}\rangle$.

only one of the points is open. $\mathfrak{C}(F^*)$ is the class of all topological spaces; here F^* denotes the three-point space in which the only non-trivial open set is one of the points. It should be noted that the class of all T_1-spaces, T_2-spaces, etc., are not classes of complete regularity. In fact, it can be shown that every class of complete regularity which contains all regular spaces must contain all T_0-spaces. (See [9].)

From the Embedding Theorem we obtain the following necessary and sufficient conditions for E-complete regularity. $X \in \mathfrak{C}(E)$ iff the class $C(X, E)$ of all continuous functions from X to E is both E-distinguishing and E-separating (or equivalently, if X admits a class of functions which is both E-distinguishing and E-separating). Note that if X is a T_0-space then an E-separating class is E-distinguishing.

The class of E-completely regular spaces is the largest class of spaces one needs to consider when dealing with algebraic properties of sets of E-valued continuous functions. If E has some algebraic operations and/or relations then the corresponding operations or relations can be defined on $C(X, E)$ by means of the usual "pointwise" definition. It turns out that every $C(X, E)$ is isomorphic to $C(X_1, E)$ where X_1 is E-completely regular. This can be proven in the following way. Let Φ be a continuous map of X onto X_1. Then Φ induces a map $\overleftarrow{\Phi}$ of $C(X_1, E)$ into $C(X, E)$ defined by

$$\overleftarrow{\Phi}(g) = g \circ \Phi \quad \text{for every} \quad g \in C(X_1, E)$$

It is always true that $\overleftarrow{\Phi}$ is an isomorphism (with respect to pointwisely defined operations) but in general it does not map $C(X_1, E)$ onto $C(X, E)$. Now we have

The Identification Theorem. *For every space X there is an E-completely regular space X_1 and a continuous map Φ of X onto X_1 such that $\overleftarrow{\Phi}$ maps $C(X_1, E)$ onto $C(X, E)$.*

The pair (X_1, Φ) is unique (up to homeomorphisms); we shall call it the E-transformation of X; X_1 will be called the E-modification of X.

The E-transformation (X_1, Φ) of X can be also characterized by the following maximality property: if f is an arbitrary continuous map of X onto an E-completely regular space Y, then there exists a continuous map g of X_1 onto Y such that $f = g \circ \Phi$.

F- and I-transformations have already been considered by Čech. If X is Hausdorff compact, then the D-modification of X is the space of components of X. (Blefko [1].)

4. E-Compact Spaces

All spaces are assumed to be Hausdorff. A space X is said to be E-compact provided that X is homeomorphic to a closed subspace of some topological product $E^{\mathfrak{m}}$ of E. The class of all E-compact spaces will be denoted by $\mathfrak{K}(E)$.

The Embedding Theorem provides us with the following criterion for E-compactness: *an E-completely regular space X is E-compact iff $C(X, E)$ is a non-extendable class* (or equivalently, *if X admits an E-non-extendable class*).

$\mathfrak{K}(I)$ is the class of all compact spaces; $\mathfrak{K}(D)$ is the class of all 0-dimensional compact spaces. The classes $\mathfrak{K}(R)$ and $\mathfrak{K}(N)$ (where R and N denote, respectively, the reals and the non-negative integers) will be discussed in Section 6.

We list the basic properties of classes of compactness. $\mathfrak{K}(E)$ contains E. Arbitrary products, intersections and closed subspaces of E-compact spaces are E-compact. If $f\colon X \to Y$ is continuous, X and Y_1 are E-compact, then $f^{-1}[Y_1]$ is also E-compact.

We have the following generalization of the compactification βX. For every $X \in \mathfrak{C}(E)$ there exists a unique E-compact extension $\beta_E X$ of X such that every continuous function $f\colon X \to Y$ where Y is E-compact, admits a continuous extension $f^*\colon \beta_E X \to Y$. $\beta_I X$ coincides with the usual βX.

If E is E_1-compact and $\mathfrak{C}(E) = \mathfrak{C}(E_1)$, then $\beta_{E_1} X \underset{\text{ext}}{\subset} \beta_E X$ for every E-completely regular X. The converse also holds true; in fact we have the following formula for $\beta_{E_1} X$:

$$\beta_{E_1} X = \{p \in \beta_E X\colon \text{there is no continuous function } f\colon \beta_E X \to E_1^* \text{ such that}$$

$$f[X] \subset E,\ f(p) \notin E\};$$

in the above, E_1^* is an E-compact superspace of E_1.

It follows from the above that $\beta_E X$ depends only upon the class of compactness of X; i.e., $\mathfrak{K}(E) = \mathfrak{K}(E_1)$ implies that $\beta_E X \underset{\text{ext}}{=} \beta_{E_1} X$ for every $X \in \mathfrak{C}(E)$.

There are various classes of compactness. If $\operatorname{cf}(\omega_\lambda) \neq \operatorname{cf}(\omega_\delta)$, then neither of the $S(\omega_\lambda)$ and $S(\omega_\delta)$ is compact with respect to the other. On the other hand, if $\operatorname{cf}(\omega_\lambda) = \operatorname{cf}(\omega_\delta)$, then $\mathfrak{K}(S(\omega_\lambda)) = \mathfrak{K}(S(\omega_\delta))$ (Blefko [1], [2]). The class $\mathfrak{K}(D)$ of all 0-dimensional compact spaces is the smallest class of compactness; the class $\mathfrak{K}(N)$ $(= \mathfrak{K}(S(\omega_0))$ is an immediate successor of $\mathfrak{K}(D)$ (i.e., if E is N-compact and N is not E-compact, then E is D-compact). However, if $\operatorname{cf}(\omega_\lambda) > \omega_0$, then there are classes of compactness between $\mathfrak{K}(D)$ and $\mathfrak{K}(S(\omega_\lambda))$. (As examples, we can take classes $\mathfrak{K}(Z_n)$; the spaces Z_n are described in Blefko, [1], [2]). There are various questions concerning classes of compactness. Does there exist an immediate successor of $\mathfrak{K}(D)$ which is contained in $\mathfrak{K}(S(\omega_\lambda))$? Other sample questions are listed in [9].

Recently, there has been an attempt to decide whether every locally E-compact space has a one-point E-compact extension. The answer to this problem in its full generality is negative. However, various sufficient conditions have been given in [9]. This approach turns out to be quite successful – numerous specific theorems can be handled by the same process. Perhaps still better results can be obtained with the aid of generalized classes of complete regularity and of compactness.

5. Generalized Classes of Complete Regularity and Compactness

These concepts were introduced by Herrlich [3], [4]. Let $\mathfrak{E}$ be a class of topological spaces. X is said to be $\mathfrak{E}$-completely regular, or $\mathfrak{E}$-compact, respectively, [in symbols, $X \in \mathfrak{C}(\mathfrak{E})$, $X \in \mathfrak{K}(\mathfrak{E})$, resp.] provided that X is homeomorphic to a subspace, closed subspace, resp., of a product of spaces from $\mathfrak{E}$. Classes of the form $\mathfrak{C}(\mathfrak{E})$ and $\mathfrak{K}(\mathfrak{E})$ will be called generalized classes of complete regularity and of compactness, respectively. This represents an ultimate generalization of our considerations; in fact, we have the following: a class of spaces is a generalized class of complete regularity (of compactness, resp.) iff it is closed under taking arbitrary products and arbitrary subspaces (arbitrary closed subspaces, resp.). Herrlich also points out that other topological operations can be used to characterize these spaces. It is important to note that the identification theorem and the existence of $\beta_{\mathfrak{E}}X$ still hold for these new classes.

Each of the classes T_1-spaces, T_2spaces, and T_3-spaces are generalized classes of complete regularity (but none are classes of complete regularity). If $\mathfrak{E}$ is the class of all $S(\omega_\alpha)$ where ω_α is an arbitrary initial ordinal, then $\mathfrak{K}(\mathfrak{E})$ is a generalized class of compactness which is not a class of compactness. There are various open questions as to whether a given generalized class of complete regularity (compactness) is a class of complete regularity (compactness). Another problem is to determine, for a given class $\mathfrak{A}$, a minimal class $\mathfrak{E}$ with $\mathfrak{A} = \mathfrak{C}(\mathfrak{E})$ ($\mathfrak{A} = \mathfrak{K}(\mathfrak{E})$).

6. R- and N-Compact Spaces

The purpose of this section is to mention some specific properties of R- and N-compact spaces. We will quote here from various authors without giving specific references. They share various common properties, but it is not known whether every 0-dimensional R-compact space is N-compact. (The converse is true.)

There are various additivity theorems. Countable additivity holds true provided that the summands are closed and every continuous R-valued (N-valued) function can be continuously extended over the union. Without the second assumption finite additivity fails for both. There are, however, some affirmative results possible in this direction: $X = X_1 \cup X_2$ is R-compact (N-compact) provided a) X_1 is R-compact (N-compact) and X_2 is compact (O-dimensional compact); b) X_1 is R-compact (N-compact) and X_2 is Lindelöf (0-dimensional Lindelöf) and both X_1 and X_2 are closed in X. The last statemet fails if X_2 is not assumed closed. It is not known whether X_1 need be assumed closed.

The last problem is open in the following particular case. Let $\mathfrak{N}$ be a class of almost disjoint subsets of N; we topologize $N \cup \mathfrak{N}$ by agreeing that points of N are isolated and neighborhoods of $A \in \mathfrak{N}$ are of the form $\{A\} \cup (A - S)$ where S is any finite subset. If $\mathfrak{N}_0$ is "small enough" then $N \cup \mathfrak{N}_0$ is N-compact (and there exist classes $\mathfrak{N}_0$

of cardinality $2^{\aleph_0}$ such that $N \cup \mathfrak{N}_0$ is N-compact). However, if we extend such a class $\mathfrak{N}_0$ to a maximal class $\mathfrak{N}_1$ of almost disjoint subsets of N, then $N \cup \mathfrak{N}_1$ is not N-compact. However, it is not known at what point in this process of enlarging the class $\mathfrak{N}_0$ to a class $\mathfrak{N}$ the space $N \cup \mathfrak{N}$ is no longer N-compact. In particular, if $\mathfrak{N}$ has only countably many more sets than $\mathfrak{N}_0$ (i.e., $N \cup \mathfrak{N}$ is obtained from $N \cup \mathfrak{N}_0$ by adding a Lindelöf space), must $N \cup \mathfrak{N}$ be N-compact? If $\mathfrak{N}$ is maximal, and we remove countably many subsets, say $\{A_n\}$, is $N \cup (\mathfrak{N} - \{A_n\})$ necessarily not N-compact? We do not know set theoretic conditions on $\mathfrak{N}$ so that $N \cup \mathfrak{N}$ is N-compact.

We next consider continuous maps. An image of an R-compact (N-compact) space under a perfect[2]) map need not be R-compact (N-compact). The problem as to whether assuming that the domain is normal alters the situation has been open for several years. It has recently been shown that the images of R-compact subsets under perfect maps of normal countably paracompact spaces are R-compact. Perhaps the resolution of the last mentioned problem will occur only when it is decided whether every normal space is countably paracompact.

There are sufficient conditions known for the R-compactness of the domain. E.g., the domain of a continuous function is R-compact (N-compact) provided that the range is hereditarily R-compact (N-compact) and the counter-images of points are compact. (In the N-compact case, the domain must be assumed 0-dimensional.)

Some results may also be obtained under the assumption that the map preserves zero-sets.

The role of Lindelöf spaces is not yet known. Every Lindelöf (0-dimensional Lindelöf) space is R-compact (N-compact). It follows that *every continuous image of a Lindelöf space is R-compact.* Does the converse hold true? Every Lindelöf space X has the following property: for every R-compact space Y, the projection of every closed subset of $X \times Y$ onto Y is R-compact. Are the Lindelöf spaces the only spaces with this property? (Analogous results hold for N-compactness and analogous questions can be stated for N-compactness under the assumption of 0-dimensionality.)

7. The Defect

This concept has arisen in connection with the estimation of exponents. We define $\exp_E X$ ($\mathrm{Exp}_E X$) to be the smallest infinite cardinal $\mathfrak{m}$ such that X is homeomorphic to a subspace (closed subspace) of $E^{\mathfrak{m}}$. The E-defect of X (in symbols, $\mathrm{def}_E X$) measures, roughly speaking, the difference between $\exp_E X$ and $\mathrm{Exp}_E X$. The exact definition (suggested in the Embedding Theorem) is as follows: $\mathrm{def}_E X$ is the smallest

[2]) A perfect map is one which is continuous, closed, and such that inverse images of points are compact.

(finite or infinite) cardinal $\mathfrak{p}$ so that there exists an E-non-extendable class for X of cardinal $\mathfrak{p}$. It is easy to see that $\mathrm{Exp}_E X = \exp_E X + \mathrm{def}_E X$. The complete product theoretic interpretation of the defect is as follows. Call a space E admissible provided that there exists a compact space E^* such that $\mathfrak{C}(E) = \mathfrak{C}(E^*)$.

Theorem. *Let E be admissible. The following conditions on an E-compact space X are equivalent*

(a) $\mathrm{def}_E X \leqq \mathfrak{p}$;

(b) *X is homeomorphic to a closed subspace of $C \times E^{\mathfrak{p}}$ where C is compact*;

(c) *for every embedding X' of X into $E^{\mathfrak{m}}$ there exists a closed embedding X'' of X into $E^{\mathfrak{m}+\mathfrak{p}}$ such that the projection of X'' onto $E^{\mathfrak{m}}$ coincides with X'.*

Various theorems concerning the preservation of E-compactness can be stated in a more comprehensive form as rules concerning the E-defect. For instance, corresponding to the theorem "*If f: $X \to Y$ is continuous, X and Y_1 are E-compact, then $f^{-1}[Y_1]$ is E-compact*", we have the rule "$\mathrm{def}_E f^{-1}[Y_1] \leqq \mathrm{def}_E X + \mathrm{def}_E Y_1$". Various results have been stated in this fashion but further work in this direction would seem to be worthwhile.

8. Structures of Continuous Functions

E-compact spaces appear in connection with the representation problem of homomorphisms (functionals) of structures of continuous functions. Suppose that E has a certain algebraic structure (i.e., some operations and/or relations are defined on E). This structure is then inherited by $C(X, E)$. We are interested in finding the form of homomorphisms of $C(X, E)$. Speaking in very rough terms, the following can frequently be shown for various cases of E. *If, for a given E, it is known that for any compact X, all $C(X, E)$ have homomorphisms of a certain form, then homomorphisms of $C(Y, E)$ will have this form iff Y is E-compact.*

These problems are studied in a series of papers entitled, "Structures of Continuous Functions". The first [10] contains a description of the present status of the investigation as well as references to other papers in this series. It should be pointed out that very few general (i.e., concerning arbitrary E) representation theorems have been obtained. However, some general proceedures have been developed.

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