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INDUCED MOVEMENTS ABOUT FIXED POINTS

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Some years ago I made a study of the set of points which are fixed under a homeomorphic transformation of a subset M of a topological space into itself. It was possible to show that the components of the complement of such a set of fixed points in M fall into groups of two types, one composed of a finite number of elements and the other of an infinite number. By putting restrictions on M or on the grouping of components or both we were able to establish certain properties of the components and of our set of fixed points: for example, if M is a sphere and there exists one group of components containing at least two elements, then our set of fixed points is a simple closed curve and our transformation T must be such that it merely interchanges the two complementary regions of this simple closed curve [5].

The question naturally arises as to the effect of lightening the conditions on the function which carries M into itself. Consideration of peripherally continuous functions in connection with locally cohesive spaces affords some interesting situations.

A function $f\colon X \to Y$ is peripherally continuous at $x \in X$ provided that if U and V are open sets about x and $f(x)$, respectively, there is an open set W such that $x \in W \subset U$ and $f[F_r(W)] \subset V$, where $F_r(W)$ is the frontier or boundary of W [1].

Theorem 1. *Let X be a Hausdorff space and let $f\colon X \to X$ be peripherally continuous, then the set F of fixed points under f is quasi-closed*[1]).

Definition. *A set X is locally cohesive provided for any point $x \in X$ and any open set U containing x, there is a canonical neighbourhood containing x which together with its closure lies in U* [2].

Theorem 2. *If X is a locally cohesive Hausdorff space and $f\colon X \to X$ is peripherally continuous, let F be the set of fixed points under f and G be a component of $X - F$, then $f(G)$ is connected.*

Definition. *A collection C_0, C_1, $C_2, \ldots$ of components of $X - F$ where X is a point set and F is the set of fixed points under $f\colon X \to X$, is said to form a finite*

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[1]) A set K in a topological space X is quasi-closed provided for any $x \in X - K$ and any open set U containing x there is an open set V, $x \in V \subset U$, such that $F_r(V) \cap K = \emptyset$.

rotation group provided (1) *the collection* $C_0, C_1, C_2, \ldots$ *contains only a finite number of components, say* $n + 1$, *and* (2) *the components of the collection may be ordered in such a way that* $f(C_0) = C_1$, $f(C_1) = C_2, \ldots, f(C_i) = C_{i+1}, \ldots$ $\ldots, f(C_n) = C_0$, *where* $f(C_j) \neq C_i$ *for* $i < j < n$; *or is said to form an infinite rotation group provided* (i) *the collection contains infinitely many components and* (ii) *it may be ordered* $C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots$ *where* $f(C_i) = C_{i+1}$ *and* $f(C_j) \neq C_i$, $i < j$ [5].

Theorem 3. *Let X be a locally cohesive regular Hausdorff space, let $f: X \to X$ be peripherally continuous, let the set of fixed points F be an inverse set and let f^{-1} either be peripherally continuous or preserve connectedness. Then, the components of $X - F$ divide up into rotation groups.*

Definition. *A point set K is said to have property S provided, for any preassigned positive number ε, K is the union of a finite number of connected set of diameter less than ε* [3].

Theorem 4. *If X is a plane continuous curve, $f: X \to X$ is peripherally continuous, and the set of fixed points F is an inverse set, and C_j is an element of a finite rotation group, then*

$$F_r(C_j) = F_r\Big(\sum_{i=0}^{n} C_i\Big) = \sum_{i=0}^{n} F_r(C_i)\,.$$

Theorem 5. *If X is a locally cohesive continuum in the plane and $f: X \to X$ is peripherally continuous and the set of fixed point F is a closed inverse set, then any element C of a finite rotation group under f of order $\geqq 2$ has property S.*

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