G. V. Krishnarao

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NONSTABLE HOMOTOPY GROUPS OF $U(n)$

G. V. KRISHNARAO

Kanpur

In [3] unstable homotopy of the orthogonal groups is partially determined; the same methods can be applied to the unitary groups. In this paper the p-primary components $\pi_m^p(U(n))$ of $\pi_m(U(n))$ are given for all odd primes p, all n, and all $m < 2p(p-1)$.

We consider the exact couple formed by the homotopy sequences of the fiberings $U(n) \to U(n+1) \to S^{2n+1}$; the spectral sequence arising out of it converges to the known homotopy groups of the infinite unitary group U. The computation of the differentials gives the results.

On tensoring the exact couple with the rationals Q, the resulting spectral sequence converges to $\pi_*(U) \otimes Q$ and it follows at once that $\pi_m^p(U(n))$ is a finite p-group except that $\pi_{2q-1}(U(n))$ has a summand Z for each q and n such that $n \geqq q$.

Let i: $\pi_m(U(n)) \to \pi_m(U(n+1))$ be induced by the inclusion.

Theorem 1. *Let m be odd and $m < 2p(p-1)$. Then $\pi_m^p(U(n)) = 0$ except that*

(i) *it is Z_p for each r and j such that $2 \leqq j \leqq r < p$ if $m = 2r(p-1) - 2 + 2j - 1$ and $j \leqq n \leqq (j-1)p$ (these groups are isomorphic under i);*

(ii) *it is Z when $m \leqq 2n - 1$.*

For completeness, (ii) which we assumed is included. The groups in (i) correspond to the differentials which annihilate the unstable elements in $\pi_m^p(S^{2j-1})$.

If m is even it is convenient to define $t = t_m$ by $2tp \leqq m < 2(t+1)p$ and to write λ_m for the least positive integer congruent to $\frac{1}{2}m + 1 \bmod (p-1)$.

Theorem 2. *Suppose m is even and $m < 2p(p-1)$. Then*

(i) $\pi_m^p(U(n)) = 0$ *for* $m < 2p$,

(ii) $\pi_m^p(U(n)) = 0$ *for* $n < \lambda_m$ *or* $n > \frac{1}{2}m$,

(iii) $\pi_m^p(U(n)) = Z_{p^{s+1}}$ *when* $\lambda_m + s(p-1) \leqq n < \lambda_m + (s+1)(p-1)$ *except that the group is only* Z_{p^s} *if* $s \geqq \frac{1}{2}m - (t_m + 1)(p-1) \geqq 1$, *where* $s = 0, 1, 2, \ldots$,

(iv) *these groups are mapped monomorphically by i when $n < m/2$.*

The groups in (iii) correspond to the non-zero differentials on $\pi_{m+1}(S^{m+1})$.

The calculations use the following unpublished results of M. G. Barratt on homotopy of spheres;

Proposition. *The group* $\pi_m^p(S^{2j-1})$ *is zero for* $m < 2p(p-1) - 2$ *except that*

(i) *it is* Z_p *generated by the stable element* α_t, *for* $j \geqq 2$ *when* $m - 2j + 1 = 2t(p-1) - 1$,

(ii) *it is* Z_p *generated by* $\theta_r^j = \{\alpha_1, \ldots, \alpha_1, \alpha_{r-j+1}\}$ (*where* $\theta_r^2 = \alpha_1\alpha_{r-1}$ *and* θ_r^j *is a j-fold Toda bracket if* $j > 2$), *when* $m - 2j + 1 = 2r(p-1) - 2$. *Here* θ_r^j *is annihilated by double suspension.*

The following result is true for all $m \geqq p$.

Theorem 3. $\pi_{2m}^p(U(n)) = Z_{p^t}$ *for* $m - p + 2 \leqq n \leqq m$, *where* p^t *is the highest power of p dividing m! These groups are isomorphic under i.*

This follows from the well-known result of Borel-Hirzebruch [2] that $\pi_{2m}(U(n)) = Z_{m!}$. In the spectral sequence the subgroup of $\pi_{2m+1}(S^{2m+1})$ generated by $m!\ i_{2m+1}$ converges to $\pi_{2m+1}(U)$, where i_{2m+1} generates $\pi_{2m+1}(S^{2m+1})$.

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INDIAN INSTITUTE OF TECHNOLOGY, KANPUR, INDIA