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Remarks on separation axioms

In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971.
pp. [265]--296.

Persistent URL: http://dml.cz/dmlcz/700579

REMARKS ON SEPARATION AXIOMS

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We are all familiar with the separation axioms T_i, i = 0, 1, ..., 5. Also, Urysohn's separation axiom $T_{2\frac{1}{2}}$ and Tychonoff's axiom $T_{3\frac{1}{2}}$ are well known. Besides these, many separation axioms have been defined for various purposes from time to time. Since the separation axioms pervade the whole of topology, it is almost impossible even to make a passing reference to all the recent work on separation axioms in a short survey. An attempt shall however, be made to give all the basic definitions of the different axioms and to summarize their inter-relations.

1. Separation Axioms Between T_0 and T_1

The first separation axiom between T_0 and T_1 was introduced by J. W. T. Youngs [92].

Youngs' axiom. *For any two distinct points x and y,* $\overline{\{x\}} \cap \overline{\{y\}}$ *is degenerate.*

Later on, several axioms between T_0 and T_1 were introduced by C. E. Aull and W. J. Thron [3]. They formulated these axioms with the help of certain basic axioms concerning the behaviour of the derived sets of points. For example, derived set of a singleton is closed or is a union of closed sets or is a union of disjoint closed sets etc. The main axioms considered by them are the following:

1. T_D: *For every point* $x \in X$, $\{x\}'$ (*that is, the derived set of* $\{x\}$) *is a closed set.*

2. T_F: *Given any point x and any finite set F such that* $x \notin F$, *either* $\{x\}$ *is weakly separated from F or F is weakly separated from* $\{x\}$.

3. T_Y: *For all* $x, y \in X$ *such that* $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ *is degenerate.*

4. T_{UD}: *For every point* $x \in X$, $\{x\}'$ *is a union of disjoint closed sets.*

5. T_{DD}: T_D + *for all* $x, y \in X$, $x \neq y$, $\{x\}' \cap \{y\}' = \emptyset$.

6. T_{YS}: *For all* $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ *is either* $\emptyset$ *or* $\{x\}$ *or* $\{y\}$.

7. T_{FF}: *For any two disjoint, finite sets* F_1 *and* F_2, *either* F_1 *is weakly separated from* F_2 *or* F_2 *is weakly separated from* F_1.

The implications between these seven axioms are the following:

$$\begin{array}{ccccccc}
T_1 & \Longrightarrow & T_{DD} & \Longrightarrow & T_D & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
 & & T_{YS} & & & & \\
 & & \Downarrow & & & & \\
T_{FF} & \Longrightarrow & T_Y \Rightarrow T_F & \Rightarrow & T_{UD} & \Rightarrow & T_0
\end{array}$$

Aull and Thron have constructed examples to show that the reverse implications do not hold in general. They have obtained various characterisations of these axioms and have shown that all of them except T_{FF} can be described in terms of the behaviour of derived sets of points. A new proof of Stone's result that every space can be made into a T_0 space by identifying indistinguishable points (that is, points having identical closures) is given. The importance of the T_D axiom is contained in the following two results which are known to be true for T_1 spaces:

1. *A space is a* T_D *space iff the derived set of each set is closed.* (In fact, as pointed out by the authors – the T_D axiom was suggested to them by a remark of C. T. Yang that the derived set of every point is closed iff the derived set of every set is closed).

2. *If the spaces* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *are both* T_D *spaces, then they are homeomorphic iff* $\mathcal{T}$ *is lattice-isomorphic to* $\mathcal{U}$.

Another separation axiom between T_0 and T_1 was introduced again by C. E. Aull [1]. This was called T_{UB} (or J_0).

T_{UB} ($\equiv J_0$): *For each* $x \in X$ *and* $M \subset X$, *M compact such that* $x \notin M$, *either* $\{x\}$ *is weakly separated from M or M is weakly separated from* $\{x\}$.

$T_{UB} \Rightarrow T_F$ and if the space is compact, then $T_{UB} \Rightarrow T_D$.

Aull and Thron [3], posed the following problem:

Does there exist a separation axiom T_α *weaker than* T_1 *such that a normal,* T_α *space is* T_4?

They remarked that none of the axioms introduced by them could serve the purpose. This problem has since been solved affirmatively in two papers.

S. M. Kim [35] has given four axioms weaker than T_1, each of which together with normality implies T_4. The axioms named as T_α, T_β, T_a, T_b by Kim are the following:

1. T_α: (i) *Let* $x \in X$, *if* $\overline{\{x\}} \neq X$ *then there exists at least one point p such that* $p \in X \sim \overline{\{x\}}$ *and* $\overline{\{p\}} = \{p\}$.

(ii) *X contains at least one point x such that* $\overline{\{x\}} = \{x\}$.

(iii) *If C and D are two disjoint closed subsets of X and have disjoint neighbourhoods, then* $\overline{\{x\}} = x$ *for each* $x \in C \cup D$.

2. T_β: (i) *Let C be a closed subset of X. If* $C \neq X$, *then there exists a non-empty closed set D such that* $C \cap D = \emptyset$.

(ii) *X contains at least one element x such that* $\overline{\{x\}} \neq X$.

(iii) *If C and D are two disjoint closed subsets of X and have disjoint neighbourhoods, then* $\overline{\{x\}} = \{x\}$ *for each* $x \in C \cup D$.

3. T_a: (i) $\overline{\{x\}} = \{x\}$ *or* $\overline{\{x\}}$ *contains at least three elements for each* $x \in X$.

(ii) *X contains at most one element x such that* $\overline{\{x\}} \neq \{x\}$.

4. T_b: (i) *If* $x \in X$ *and* $\overline{\{x\}} \neq X$, *then there exists at least one point p such that* $p \in X \sim \overline{\{x\}}$, *and* $\overline{\{p\}} = \{p\}$.

(ii) *If C, D are two disjoint closed subsets of X and have disjoint neighbourhoods, then* $\overline{\{x\}} = \{x\}$ *for each* $x \in C \cup D$.

(iii) *X contains at most one element x such that* $\overline{\{x\}} \neq \{x\}$.

All the above four axioms are based on the observation that in a T_1 space $\overline{\{x\}} = \{x\}$ for all $x \in X$.

Kim proved that a normal space which is either T_α or T_β or T_a or T_b is a T_4 space. None of these axioms is preserved under a strengthening of the topology. In fact, Kim proved that there exists no separation axiom which is preserved under a strengthening of the topology and is weaker than T_1 such that together with normality it could imply T_4.

Y. C. Wu and S. M. Robinson [90] introduced two new separation axioms, namely, strong T_0 and strong T_D.

Strong T_0: *For each* $x \in X$, $\{x\}'$ *is a union of a family of closed sets such that the intersection of non-empty members of the family is empty and at least one of the non-empty members is compact.*

Strong T_D: *For each* $x \in X$, $\{x\}'$ *is a finite union of closed sets, the intersection of the non-empty members of which is empty.*

Wu and Robinson proved that a normal space which is either strong T_0 or strong T_D is T_4.

W. J. Thron [80] asked whether the product of T_D spaces is again a T_D space. Wu and Robinson proved that this is never true in the infinite case. They proved the following result:

If $\{X_\alpha: \alpha \in \Lambda\}$ *is an infinite family of* T_D *spaces which are not* T_1, then $\Pi\{X_\alpha: \alpha \in \Lambda\}$ *is not a* T_D *space.*

It follows that there is no possibility of introducing a separation axiom between T_D and T_1 that is inherited by arbitrary products.

There is another separation axiom – the R_0 axiom weaker than T_1 but independent of T_0. This was first defined by N. A. Shanin [60] and was rediscovered by

A. S. Davis [16] who gave several interesting characterisations of it and proved that $T_1 = R_0 + T_0$.

R_0: *For any closed set F and a point $x \notin F$, F can be weakly separated from $\{x\}$.*

A space is R_0 iff for each open set G, and each $x \in G$, $\overline{\{x\}}$ is contained in G. Also, a space is R_0 iff for any two points x and y with $x \neq y$, either $\overline{\{x\}} = \overline{\{y\}}$ or $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$. This axiom has a certain symmetry in the sense that in such a space $x \in \overline{\{y\}}$ iff $y \in \overline{\{x\}}$.

S. A. Naimpally [47] has proved the following:

A space $(X, \mathscr{T})$ is R_0 iff there exists a compatible quasi-uniformity $\mathscr{U}$ on X such that for each $x \in X$ and $U \in \mathscr{U}$, there exists a symmetric $V \in \mathscr{U}$ such that $V[x] \subseteq U[x]$; that is, the family $\{V[x]\}$, as V runs through symmetric members of $\mathscr{U}$, forms a local basis at x for each $x \in X$.

M. G. Murdeshwar and S. A. Naimpally [45] have proved the following results:

1. *A quasi-uniform space $(X, \mathscr{U})$ is R_0 iff for each $x \in X$, $\overline{\{x\}} = \bigcap_{U \in \mathscr{U}} U[x]$.*
2. *A quasi-uniform space $(X, \mathscr{U})$ is R_0 iff $\bigcap_{U \in \mathscr{U}} U$ is symmetric.*
3. *X is R_0 iff there exists a compatible quasi-uniformity $\mathscr{U}$ such that for each $x \in X$, $U \in \mathscr{U}$ there is $V \in \mathscr{U}$ such that $V^{-1}[x] \subset U[x]$.*
4. *X is R_0 iff there exists a compatible quasi-uniformity such that*

$$\mathscr{T} = \mathscr{T}(\mathscr{U}) = \mathscr{T}(\mathscr{U}^{-1}).$$

2. Separation Axioms Between T_1 and T_2

During the last few years properties between T_1 and T_2 have been discussed by many authors. The following two may be mentioned in particular.

(1) Compact sets are closed.

(2) Convergent sequences have unique limits.

Spaces satisfying (1) have been studied by E. Halfar [24], A. J. Insell [32], N. Levine [41], C. E. Aull [4] and the spaces satisfying (2) were studied by H. F. Cullen [13], M. G. Murdeshwar and S. A. Naimpally [46] and P. Slepian [69]. Most of the results of these papers were overlapping and a systematic treatment was given by A. Wilansky [88].

A. Wilansky called the spaces satisfying (1) kc and those satisfying (2) us. Halfar showed first of all that in a first axiom space, kc $\Leftrightarrow T_2$. Cullen proved that kc $\Rightarrow$ us and that in a first axiom space us $\Leftrightarrow T_2$. Thus we have

$$T_2 \Rightarrow \text{kc} \Rightarrow \text{us} \Rightarrow T_1. \quad \text{Also, in a first axiom space, } T_2 \Leftrightarrow \text{kc} \Leftrightarrow \text{us}.$$

Wilansky proved that in a non first axiom space the reverse implications do not hold even if the space be compact. It can be proved that in a sequential space kc $\Leftrightarrow$ us. Aull called kc spaces J_1'. He proved that $J_1' \equiv J_1 \equiv JJ_1$ where J_1 and JJ_1 are as below:

J_1: *If* $M \subset X$ *is compact,* $x \notin M$, *then* $\{x\}$ *is weakly separated from* M *and* M *is weakly separated from* $\{x\}$.

JJ_1: *If* M *and* N *are compact sets such that* $M \cap N = \emptyset$, *then* M *is weakly-separated from* N *and* N *is weakly-separated from* M.

Wilansky proved that for locally-compact spaces (that is, every neighbourhood of a point contains a compact neighbourhood of that point), $T_2 \Leftrightarrow$ kc.

As a matter of fact, kc spaces were also considered long back by E. Hewitt [29] and R. Vaidyanathaswami [83]. It was proved by A. Ramanathan [53] that a kc compact space is maximal compact and minimal kc and that every maximal compact space is kc. Thus a compact space is maximal compact iff it is kc. Since a compact T_2 space is maximal compact but not conversely, it follows (as pointed out by Wilansky) that kc spaces have this advantage over T_2, that the converse also holds if T_2 be replaced by kc. It was proved by Wilansky that the one point compactification X^* of a kc space is us. He also proved that if X be a kc space than X is kc iff X^* is a k-space (that, is a subset A such that $A \cap K$ is closed for all closed and compact K is itself closed). Concerning us spaces he proved that if X is us then X^* is us iff every convergent sequence has a relatively-compact subsequence.

us spaces were studied by M. G. Murdeshwar and S. A. Naimpally under the name Semi-Hausdorff spaces. They obtained many basic properties of these spaces. S. P. Franklin [22] gives an example of a Fréchet, us, non T_2 space. An example can be given of a Fréchet, kc, non T_2 space.

Some more axioms between T_1 and T_2 namely, S_1 and S_2, have been introduced by Aull [2].

S_1: us + *every convergent sequence has a subsequence without side points* (*A point* y *is a side point of a sequence* $\{x_n\}$ *if* y *is an accumulation point of the set of values* $\{x_n\}$ *but no subsequence of* $\{x_n\}$ *converges to* y).

S_2: us + *no convergent sequence has a side point.*

We have,

$$T_2 \Rightarrow \text{kc} \Rightarrow S_2 \Rightarrow S_1 \Rightarrow S_0 \quad (\equiv \text{us}) \Rightarrow T_1 .$$

Another axiom between T_1 and T_2 has been discussed by T. Soundararajan under the name weakly-Hausdorff spaces.

Weakly-Hausdorff. *Every point is the intersection of regularly closed sets.*

Aull [4] introduced another axiom, namely the T_{AB} axiom, between T_1 and T_2.

T_{AB}: T_1 + *for any two disjoint compact sets M and N, either M is weakly-separated from N or N is weakly-separated from M.*

We have, $T_2 \Rightarrow kc \Rightarrow T_{AB} \Rightarrow us \Rightarrow T_1$.

Consider the following axiom:

Every two points with disjoint closures can be strongly separated.

The above axiom was first mentioned by C. T. Yang [91]. Yang named this axiom T_2'. He encountered it in connection with the study of paracompact spaces. Later on, A. S. Davis named this axiom R_1 and proved that $R_1 + T_1 = T_2$. Murdeshwar and Naimpally studied it in more detail and pointed out that R_1 was independent of both T_0 and T_1. It can also be shown that R_1 is independent of kc. However, it was proved by Murdeshwar and Naimpally that in an R_1 space, $T_2 \Leftrightarrow kc \Leftrightarrow us$. They obtained several other properties of R_1 spaces.

L. C. Robertson and S. P. Franklin [55] characterized the spaces in which 0-sequences are eventually constant and remarked that such spaces occupy a position between T_0 and T_2 in general, and for R_0 or homogeneous spaces, between T_1 and T_2. A. Wilansky points out, however, that such spaces are strictly between T_1 and us.

There may be several other conditions between T_1 and T_2. We have recently started a study of some of these. These are the following:

(i) ckc – *countable, compact sets closed.*

(ii) akc – *almost compact sets closed.*
(*A space is said to be almost compact if every open cover has a finite subfamily whose closures cover the space.*)

(iii) cakc – *countable, almost compact sets closed.*

(iv) nkc – *nearly compact sets closed.*
(*A space is said to be nearly compact if every regular open cover has a finite subcover.*)

(v) cnkc – *countable, nearly compact sets closed.*

It follows that all the conditions (i)–(v) above are between T_1 and T_2 in view of the following:

(i) *Every compact space is almost compact.*

(ii) *Every nearly compact space is almost compact.*

(iii) *Every almost compact subset of a Hausdorff space is closed.*

(iv) *If all countable, compact sets are closed then sequential limits are unique.*

The following implications are obvious:

$$\begin{array}{ccccccccc} T_2 \Rightarrow & \text{akc} & \Rightarrow & \text{nkc} & \Rightarrow & \text{kc} & & & \\ & \Downarrow & & \Downarrow & & \Downarrow & & & \\ & \text{cakc} & \Rightarrow & \text{cnkc} & \Rightarrow & \text{ckc} & \Rightarrow & \text{us} \Rightarrow & T_1 \end{array}$$

As we have already said above, the work on these axioms has just been started and is in a very preliminary stage. Examples are still to be constructed to show that all these axioms are distinct. However, akc can be shown to be strictly between T_2 and kc. We do hope that a detailed study of these axioms will prove useful and will yield some beautiful and fruitful results.

3. Separation Axioms Between T_2 and T_5

A new separation axiom between T_2 and T_5 was introduced by C. E. Aull [1], namely, the T_{BB} axiom.

T_{BB}: *Any two sets M and N which are separated by compact sets, (that is, there exist compact sets M^*, N^*, such that $M \subset M^*$, $N \subset N^*$, $M^* \cap N = \emptyset$, $M \cap N^* = \emptyset$) can be strongly separated.*

Aull proved that in general, $T_5 \Rightarrow T_{BB} \Rightarrow T_2$, for compact spaces $T_{BB} \Leftrightarrow T_5$ and for spaces satisfying the second axiom of countability, $T_2 \Leftrightarrow T_{BB}$. As a consequence of these two results, it follows that there exist T_2, T_3 and T_4 spaces which are not T_{BB} and there exist T_{BB} spaces which are not T_3, T_4 or T_5. He also proved that a T_2 space with a σ-locally finite base is T_{BB}.

J. P. Thomas [79] defines a new separation axiom, namely T_{2b} and shows that it is between $T_{2\frac{1}{2}}$ and T_3.

T_{2b}: *A space $(X, \mathscr{T})$ is said to be a T_{2b} space if $(X, \mathscr{T}_*)$ is T_1 where $\mathscr{T}_*$ is the least upper bound of all regular topologies on X coarser than $\mathscr{T}$.*

Thomas remarks that he knows of no characterisation of T_{2b} spaces orther than the defining one.

Apart from the T_2' ($\equiv R_1$) axiom already mentioned, C. T. Yang introduced two more axioms namely LT_4 and T_3'.

T_3': *For each point x and each neighbourhood U of $\overline{\{x\}}$, there is a neighbourhood V of x such that $\overline{V} \subset U$.*

LT_4: *Every point has a neighbourhood whose closure is normal.*

The following implications are obvious:

$$\begin{array}{ccccc} T_4 & \Rightarrow & T_3 & \Rightarrow & T_2 \\ \Downarrow & & \Downarrow & & \Downarrow \\ LT_4 & \Rightarrow & T_3' & \Rightarrow & T_2' \end{array}$$

Yang proved that in a paracompact space, T_2', T_3', LT_4 and T_4 are all equivalent.

Recently, J. Mack [43] has introduced the concept of m-normality. This is defined as follows:

$\mathfrak{m}$-normal. *For an infinite cardinal* $\mathfrak{m}$, *a space is said to be* $\mathfrak{m}$-*normal if each pair of disjoint closed sets, one of which is a regular* $G_{\mathfrak{m}}$-*set (that is, the intersection of atmost* $\mathfrak{m}$ *closed sets whose interiors contain* A), *can be strongly separated.*

Following are some of the results proved by J. Mack concerning $\mathfrak{m}$-normal spaces:

1. *Every* $\mathfrak{m}$-*paracompact space is* $\mathfrak{m}$-*normal.*
2. *A Hausdorff,* $\mathfrak{m}$-*normal space, every point of which has a neighbourhood basis of cardinality* $\leqq \mathfrak{m}$, *is regular.*
3. *A Hausdorff,* $\mathfrak{m}$-*normal space having cardinality* $\leqq \mathfrak{m}$, *is regular.*
4. *If an* $\aleph_0$-*normal space is either countable or satisfied the first axiom of countability, then it is regular.*
5. *Every closed, continuous image of an* $\mathfrak{m}$-*normal space is* $\mathfrak{m}$-*normal.*

M. H. Stone [74] discussed semi-regular spaces and showed that even a Hausdorff semi-regular space may fail to be regular. This poses the problem of discovering, what restrictions upon semi-regular spaces imply regularity. M. K. Singal and S. P. Arya [65] have introduced a separation axiom, namely almost-regularity which together with semi-regularity implies regularity.

Almost-regularity. *For each regularly closed set* A *and a point* $x \notin A$, *there exist disjoint open sets* U *and* V *such that* $A \subset U$, $x \in V$.

Examples have been constructed to show that almost-regularity is independent of semi-regularity. Also, it has been shown that for Hausdorff spaces this axiom occupies a position between Urysohn and T_3 spaces. An example is given of an almost-regular, T_1 space which is not Urysohn. It is known that every regular, T_0 space is T_1. However, it has been shown that an almost-regular T_0 space may not be T_1. Several characterisations of almost-regular spaces have been obtained. Many other properties concerning subsets, product etc. of almost-regular spaces have been obtained.

M. K. Singal and S. P. Arya [65] introduce another separation axiom, namely weak-regularity which is weaker than almost-regularity even, but becomes equivalent to it for T_1 spaces. This is defined as below:

Weak-regularity. *Every weakly-separated pair consisting of a regularly-closed set and a singleton can be strongly separated.*

This is equivalent to the condition: *For every point* x *and every regularly-open set* U *containing* $\overline{\{x\}}$, *there is an open set* V *such that* $x \in V \subset \overline{V} \subset U$.

While every weakly-regular, T_1 space is almost-regular, example is given of a non-T_1, weakly regular space which is not almost-regular.

Almost-regular spaces have also been studied in [67]. We list below some of the results obtained in this paper:

1. *An almost-regular space is* AR-*closed (that is, closed in every almost-regular space in which it can be embedded) iff it is subcompact (that is, every open cover* $\mathcal{G}$ *such that for each* $G \in \mathcal{G}$*, there exists* $H \in \mathcal{G}$: $\bar{G} \subset H$*, has a finite subcover).*

2. *An almost-regular space is minimal almost-regular if and only if every regular filter base with a unique adherent point is convergent.*

3. *An almost-regular space is minimal almost-regular if and only if it is minimal regular.*

4. *Every minimal almost-regular space is pseudo-compact (or* AR-*closed).*

Again, M. K. Singal and S. P. Arya [66] introduce more separation axioms which are related to normality and complete-regularity in the same way as almost-regularity is related to regularity. These axioms are named as, almost-normality and almost-complete regularity defined as follows:

Almost-normality. *For every pair of disjoint sets* A *and* B *one of which is closed and the other is regularly-closed, there exist open sets* U *and* V *such that* $U \cap V = \emptyset$, $A \subset U$, $B \subset V$.

Almost-complete-regularity. *For every regularly-closed set* A *and a point* $x \notin A$*, there is a continuous function* f *on* X *into the closed interval* $[0, 1]$ *such that* $f(x) = \{1\}$, $f(A) = \{0\}$.

An example has been given to show that, not every almost-normal space is normal. Every almost-normal, T_1 space is almost-regular. But in general, almost-normality does not necessarily imply almost-regularity. The concept of semi-normality is also introduced in the same paper as follows:

Semi-normality. *Every closed set has a base consisting of regularly open sets for the open sets containing it.*

Examples have been constructed to show that semi-normality is independent of almost-normality. However it has been shown that almost normal + semi-normal $\Leftrightarrow$ normal. Also, examples are being constructed to show that semi-normality is independent of semi-regularity. But every semi-normal, T_1 space is semi-regular.

In the above paper, several characterizations of almost-normality have been obtained. It has been proved that every weakly-regular, paracompact space is almost-normal. The following result for almost-normal spaces, analogous to the well known Urysohn's Lemma for normal spaces has been obtained. A space is almost-normal if and only if for every pair of disjoint sets A and B, one of which is closed and the other is regularly closed, there exists a continuous function f on X into $[0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Every almost-completely-regular space is almost-regular. Also, every almost-normal, T_1 space is almost-completely-regular. But as already mentioned an almost-normal space is not necessarily almost-regular and hence not almost-completely-regular. Several properties of almost-completely-regular spaces have been obtained. It has been shown that every almost-regular, almost-normal space is almost-completely-regular. It is well known that every regular, normal space is completely regular. This result is sharpened as follows:

Every regular, almost-normal space is completely-regular.

4. Separation Axioms in Ordered Spaces

A topological ordered space $(X, \mathscr{T}, \leqq)$ is a set X endowed with both a topology $\mathscr{T}$ and an order relation $\leqq$. The study of order relations in topological spaces was initiated by L. Nachbin. Most of his results can be found in his monograph "Topology and Order" which is an English translation of his Portuguese monograph published in 1950. As regards separation axioms in topological ordered spaces, Nachbin's main interest was in generalizing the basic facts of the theory of normal spaces. Nachbin's results were extended and generalized by L. E. Ward [86] who defined several types of order normality and regularity for spaces more general than those of Nachbin and improved several of his results. A systematic study of separation axioms for topological ordered spaces has also been done recently by S. D. McCartan [44]. He has studied T_i axioms $(i = 1, 2, 3, 4)$ in topological ordered spaces and has called them T_i order axioms $(i = 1, 2, 3, 4)$. He has shown that each T_i order axiom is successively stronger than the T_{i-1} order axiom and is also stronger than the T_i axiom $(i = 1, 2, 3, 4)$ for general topological spaces and has constructed several examples to show that the converse statement is not necessarily true.

Let $(X, \mathscr{T})$ be a space and let "$\leqq$" be a binary relation in X. "$\leqq$" is called a *pre-order* if it is reflexive and transitive. An *order* (or a partial order) is a pre-order which is also antisymmetric. Throughout the sequel, it will be assumed that $(X, \mathscr{T}, \leqq)$ (Usually denoted as X only) is a topological ordered space where "$\leqq$" is any binary relation in X. For any $x, y \in X$, $x \parallel y$ means that $x \nleqq y$ and $y \nleqq x$, $[x, \rightarrow]$ (resp. $[\leftarrow, x]$) denotes the set of elements $y \in X$ such that $x \leqq y$ (resp. $y \leqq x$), $i(A)$ denotes the set $\{[a, \rightarrow]: a \in A\}$ and $d(A)$ is the set $\{[\leftarrow, a]: a \in A\}$. A is said to be *increasing* if $A \supset i(A)$ and *decreasing* if $A \supset d(A)$.

We first list a number of separation axioms introduced by Ward. Some of these reduce to those introduced by Nachbin if the order considered is a pre-order.

MR (Monotone regularity). *For each $x \in X \sim F$, where F is closed and increasing (decreasing), there are disjoint open sets U and V such that U is decreasing (increasing), V is increasing (decreasing) and $x \in U$, $F \subset V$.*

MR′. *Whenever* x, F *are as above, there exists a continuous function* f *on* X *into* $[0, 1]$ *such that* $f(x) = 0$ $(f(x) = 1)$ *and* $f(F) = \{1\}$ $(f(F) = \{0\})$.

SR (Strong regularity). *Whenever* F *is a closed set,* $x \in X \sim F$ *such that* $d(x) \cap F = \emptyset$, $i(x) \cap F = \emptyset$, *there are disjoint open sets* U *and* V *such that* U *is decreasing (increasing),* V *is increasing (decreasing) and* $x \in U$, $F \subset V$.

SR′. *Whenever* x *and* F *are as above, there is a continuous function* f *on* X *into* $[0, 1]$ *such that* $f(x) = 0$ $(f(x) = 1)$ *and* $f(F) = \{1\}$ $(f(F) = \{0\})$.

OCR (Order complete regularity). *Whenever* F *is a closed set and* $x \in X \sim \sim F$, *there are continuous functions* f *and* g *on* X *into* $[0, 1]$ *such that* $f(x) = 1$, $g(x) = 0$ *and for* $t \in F$, *either* $f(t) = 0$ *or* $g(t) = 1$.

MN (Monotone normality). *Whenever* F_0 *and* F_1 *are disjoint closed sets such that* F_0 *is decreasing,* F_1 *is increasing, there are disjoint open sets* U_0 *and* U_1 *such that* $F_0 \subset U_0$, $F_1 \subset U_1$ *and* U_0 *is decreasing and* U_1 *is increasing.*

MN′. *Whenever* F_0, F_1 *are as above, there is a continuous function* f *on* X *into* $[0, 1]$ *such that* $f(F_0) = \{0\}, f(F_1) = \{1\}$.

SN (Strong normality). *Whenever* F_0 *and* F_1 *are disjoint closed sets such that* F_0 *is decreasing (or* F_1 *is increasing) there are disjoint open sets* U_0 *and* U_1 *such that* $F_0 \subset U_0$, $F_1 \subset U_1$ *and* U_0 *is decreasing,* U_1 *is increasing.*

SN′. *Whenever* F_0, F_1 *are as above, there is a continuous function* f *on* X *into* $[0, 1]$ *such that* $f(F_0) = \{0\}, f(F_1) = \{1\}$.

Relationships between various axioms are as follows:

$$\begin{array}{ccccccc} \mathrm{SN}' & \Rightarrow & \mathrm{SN} & \quad & \mathrm{SR}' & \Rightarrow & \mathrm{SR} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{MN}' & \Rightarrow & \mathrm{MN} & & \mathrm{MR}' & \Rightarrow & \mathrm{MR} \end{array}$$

Also, when the order relation is transitive and semi-continuous ("$\leqq$" is lower (upper) semi-continuous, if $[\leftarrow, x]$ $([x, \rightarrow])$ is a closed set for each $x \in X$ and "$\leqq$" is semi-continuous if it is both upper and lower semi-continuous), then,

$$\mathrm{SN} \Rightarrow \mathrm{SR}\,, \qquad \mathrm{MN} \Rightarrow \mathrm{MR}\,,$$

$$\mathrm{SN}' \Rightarrow \mathrm{SR}'\,, \qquad \mathrm{MN}' \Rightarrow \mathrm{MR}'\,.$$

If "$\leqq$" is the trivial relation ($x \leqq y$ iff $x = y$) then all four types of normality are equivalent to normality in general topological spaces. Similarly MR and SR reduce to regularity and MR′, SR′ and OCR reduce to complete regularity. It is shown that Urysohn's lemma holds for monotone and strong normality, so that $\mathrm{MN} \Leftrightarrow \mathrm{MN}'$, $\mathrm{SN} \Leftrightarrow \mathrm{SN}'$. Ward proved by examples that these are the only equivalences even when the order relation is continuous (the graph of the order is a closed subset of $X \times X$).

The following diagramme indicates various relationships more clearly.

$$\begin{array}{ccccccccc}
\mathrm{OCR} & \nleftarrow & \mathrm{SN}' & \rightleftarrows & \mathrm{SN} & \overset{\rightarrow}{\nleftarrow} & \mathrm{SR}' & \overset{\rightarrow}{\nleftarrow} & \mathrm{SR} \\
 & & \downarrow\not\uparrow & & \downarrow\not\uparrow & & \downarrow\not\uparrow & & \downarrow\not\uparrow \\
 & & \mathrm{MN}' & \rightleftarrows & \mathrm{MN} & \overset{\rightarrow}{\nleftarrow} & \mathrm{MR}' & \overset{\rightarrow}{\nleftarrow} & \mathrm{MR}
\end{array}$$

Next we state the analogues of Urysohn's Lemma, which for monotone normality was obtained by Nachbin in the case when "$\leqq$" is a preorder and by Ward in the case when "$\leqq$" is any binary relation.

Urysohn's Lemma. *A space X is monotone normal (strongly normal) iff for any two disjoint closed subsets F_0 and F_1 of X where F_0 is decreasing and (or) F_1 is increasing, there exists on X into $[0, 1]$ a continuous function f such that $f(F_0) = \{0\}$, $f(F_1) = \{1\}$, $f(x) \leqq f(y)$ if $x \leqq y$.*

Tietze's Extension Theorem. *Let X be monotone normal and F be a closed subset of X. Let f be a bounded continuous real valued function on F: $f(x) \leqq f(y)$ when $x \leqq y$. Let $A(z) = \{x \in F: f(x) \leqq z\}$ and $B(z) = \{x \in F: f(x) \geqq z\}$ where z is any real number. Then the function f can be extended to X in such a way that it becomes a continuous, bounded real-valued function such that $f(x) \leqq f(y)$ for $x \leqq y$ if and only if $z < z' \Rightarrow D(A(z)) \cap I(B(z)) = \emptyset$ where $D(A(z))$ $(I(B(z)))$ is the smallest closed decreasing (increasing) subset containing $A(z)$ $(B(z))$.*

Recently, the following generalization of the Urysohn's lemma mentioned above has been obtained by Y. F. Lin [42].

Let X be a space equipped with a continuous relation "$\leqq$" such that X is strongly normal, F is an increasing (decreasing) closed subset of X, and $\{U_\alpha: \alpha \in \Lambda\}$ a locally finite family of increasing (decreasing) open sets in X such that $\bigcup\{U_\alpha: \alpha \in \Lambda\} \supset F$; then there exists a family $\{f_\alpha: \alpha \in \Lambda\}$ of continuous functions on X into $[0, 1]$ such that:

(i) *If $x \leqq y$ then $\sum_{\alpha\in\Lambda} f_\alpha(x) \leqq \sum_{\alpha\in\Lambda} f_\alpha(y)$.*

(ii) $\sum_{\alpha\in\Lambda} f_\alpha(x) = 1$ $\left(\sum_{\alpha\in\Lambda} f_\alpha(x) = 0\right)$ *for all $x \in F$.*

(iii) *For each $\alpha \in \Lambda$, $f_\alpha(x) = 0$ $(f_\alpha(x) = 1)$ for all $x \in X - U_\alpha$.*

Next, we summarize the separation axioms introduced by McCartan. Throughout in the following, the binary relation considered is an order relation.

T_1-ordered. *A topological ordered space X is said to be upper (lower) T_1-ordered if for each pair of elements a, b such that $a \nleqq b$ in X, there exists a decreasing (increasing) neighbourhood W of $b(a)$ such that $a \notin W$ $(b \notin W)$. X is said to be T_1-ordered if it is both lower and upper T_1-ordered.*

The concept of T_1-order coincides with the concepts of semi-continuous partial order and semi-closed partial order of Ward and Nachbin respectively in view of the following result of McCartan:

For a topological ordered space X, the following are equivalent:

(i) *X is lower (upper) T_1-ordered.*

(ii) *For each pair $a \nleqq b$ in X there exists an open set U containing $a(b)$ such that $x \nleqq b$ $(a \nleqq x)$ for all $x \in U$.*

(iii) *For each $x \in X$, $[\leftarrow, x]$ $([x, \rightarrow])$ is closed.*

(iv) *When the net $\{x_\alpha : \alpha \in \Lambda\}$ converges to a and $x_\alpha \leqq b$ $(b \leqq x_\alpha)$ for each $\alpha \in \Lambda$, then $a \leqq b$ $(b \leqq a)$.*

T_2-ordered. *A topological ordered space X is said to be T_2-ordered (or Hausdorff-ordered or H-ordered) if for each pair of elements a, b in X; $a \nleqq b$, there exist disjoint neighbourhoods U, V of a, b respectively such that U is increasing, V is decreasing.*

It should be noted that the concept of T_2-order coincides with the concepts of continuous partial order and closed partial order of Ward and Nachbin respectively.

McCartan has proved the following result:

For a topological ordered space X, the following are equivalent:

(a) *X is T_2-ordered.*

(b) *For each pair of elements a, b in X: $a \nleqq b$, there exist open sets U and V containing a and b respectively such that $x \in U$, $y \in V$ together imply $x \nleqq y$.*

(c) *The graph of the order of X is a closed subset of the product space $X \times X$.*

(d) *If nets $\{x_\alpha : \alpha \in \Lambda\}$ and $\{y_\alpha : \alpha \in \Lambda\}$ in X where $x_\alpha \leqq y_\alpha$ for each $\alpha \in \Lambda$, converge to a, b respectively, then $a \leqq b$.*

T_3-ordered. *A topological ordered space is called T_3-ordered if it is T_1-ordered and monotone regular (named as regularly ordered by McCartan) in the sense of Ward.*

McCartan proves the following result regarding regularly ordered spaces:

For a topological ordered space X, the following are equivalent:

(a) *X is regularly ordered.*

(b) *For each* $x \in X$ *and each increasing (decreasing) open neighbourhood* U *of* x, *there exists an increasing (decreasing) neighbourhood* V *of* x *such that* $\overline{V} \subset U$.

(c) *If, when a net* $\{x_\alpha : \alpha \in \Lambda\}$ *is residually contained in each increasing (decreasing) neighbourhood of an element* a *and a net* $\{y_\alpha : \alpha \in \Lambda\}$ *is residually contained in each decreasing (increasing) neighbourhood of closed decreasing (increasing) set* F, $x_\alpha \leqq y_\alpha$ *for each* $\alpha \in \Lambda$, *then* $a \in F$.

T_4-**ordered.** *A topological ordered space* X *is said to be* T_4*-ordered if it is* T_1*-ordered and monotone normal (named as normally ordered by McCartan) in the sense of Ward.*

The results proved by McCartan regarding normally ordered spaces is as below:

In a topological space X *the following are equivalent:*

(a) X *is normally ordered.*

(b) *For each increasing (decreasing) closed set* F *and each increasing (decreasing) open set* U *containing* F, *there exists an increasing (decreasing) neighbourhood* V *of* F *such that* $\overline{V} \subseteq U$.

(c) *If, when a net* $\{x_\alpha : \alpha \in \Lambda\}$ *is residually contained in each increasing neighbourhood of an increasing closed set* F_1 *and a net* $\{y_\alpha : \alpha \in \Lambda\}$ *is residually contained in each decreasing neighbourhood of a closed, decreasing set* F_2, *where either* $F_1 \cap F_2 = \emptyset$ *or* $F_1 = F_2$, $x_\alpha \leqq y_\alpha$ *for each* $\alpha \in \Lambda$, *then* $F_1 = F_2$.

McCartan remarks that further order separation axioms, known as the *strong* T_i *order separation axioms* may be obtained by replacing the word neighbourhood by open neighbourhood in the definitions of T_i order separation axioms ($i = 1, 2, 3, 4$) described above. Obviously, if X is strongly T_i-ordered then X is T_i-ordered ($i = 1, 2, 3, 4$).

S. P. Franklin and R. H. Sorgenfrey [23] consider a relation $T \subseteq X \times Y$ for any two spaces X and Y. The relation T is said to be from X to Y if the domain of T is X. T will be called upper semi-continuous if for each point x of the domain of T and for each neighbourhood V of $T(x)$, there exists a neighbourhood U of x such that $T(U) \subset V$. T will be called closed if it is a closed subset of $X \times Y$. T will be said to be image closed if each $T(x)$ is closed. Let T' be the relation defined by $T'(x) = \overline{T(x)}$ and let $\overline{T}$ be the closure of T in $X \times Y$. The authors also introduce the notion of net space which is defined as below:

Let Δ *be a directed set and let* $p \notin \Delta$. *Define a topology for* $X = \Delta \cup \{p\}$ *by letting each point of* Δ *be isolated and taking as a base at* p *all sets of the form* $S \cup \{p\}$ *where* S *is a final segment in* Δ. *When equipped with this topology,* X *will be called a net space of* Δ. *Each net space has at most one accumulation point.*

With these definitions, the following characterizations of regularity and normality are obtained:

1. *If a space Y is regular, then for each net space X and every upper semi-continuous, image closed relation T on X into Y, T is closed.*

2. *A space Y is normal iff for each net space X and each upper semicontinuous relation T on X into Y, T′ and $\overline{T}$ are upper semi-continuous.*

5. Separation Axioms in Bitopological Spaces

The study of bitopological spaces was initiated by J. C. Kelly [34]. He was motivated by the following:

A function $d: X \times X \to R$ is called a quasi-pseudo-metric (q-p-*metric*) *on X if,*

(i) $d(x, y) \geqq 0$,
(ii) $d(x, x) = 0$,
(iii) $d(x, z) \leqq d(x, y) + d(y, z)$.

With every q-p-metric d on X, there is associated another q-p-metric d^* on X defined by $d^*(x, y) = d(y, x)$. The open d-spheres and d^*-spheres will form bases for topologies $\mathscr{T}$ and $\mathscr{T}^*$ on X respectively. Now, if one studies X with two topologies $\mathscr{T}$ and $\mathscr{T}^*$, some of the symmetry of the classical metric situation is regained and one can obtain systematic generalizations of standard results such as Urysohn's Lemma, Urysohn's metrization theorem, Tietze's extension theorem and the Baire category theorem etc. This led to the consideration of any set X equipped with any two topological structures $\mathscr{T}_1$ and $\mathscr{T}_2$ which was called a bitopological space. After the publication of Kelly's paper, many authors have shown interest in the study of such spaces, for example, E. P. Lane, W. J. Pervin, P. Fletcher, Y. W. Kim, C. W. Patty, H. B. Hoyle III, M. G. Murdeshwar, S. A. Naimpally etc. We shall be concerned here with the work done on separation axioms in bitopological spaces.

A separation axiom T in a bitopological space is generally denoted by p-T (pairwise-T). Pairwise regular, pairwise normal and pairwise Hausdorff spaces are defined by J. C. Kelly as follows:

p-**regular.** *For each $x \in X$ and each $\mathscr{T}_1$-open ($\mathscr{T}_2$-open) set U such that $x \in U$, there exists a $\mathscr{T}_1$-open ($\mathscr{T}_2$-open) set V: $x \in V \subseteq \mathscr{T}_2(\mathscr{T}_1) - \mathrm{cl}\,(V) \subseteq U$.*

p-**normal.** *For every pair of disjoint sets A and B such that A is $\mathscr{T}_1$-closed, B is $\mathscr{T}_2$-closed, there exist disjoint sets U and V such that U is $\mathscr{T}_2$-open, V is $\mathscr{T}_1$-open and $A \subseteq U$, $B \subseteq V$.*

p-**Hausdorff.** *For every pair of distinct points x and y, there exist disjoint sets U and V such that $x \in U$, $y \in V$, U is $\mathscr{T}_1$-open, V is $\mathscr{T}_2$-open.*

The concept of p-regularity and p-Hausdorff may be compared with those of "coupling" and "consistency" of two topologies introduced by J. D. Weston [87]. These are defined as follows:

$\mathscr{T}_1$ is said to be coupled to $\mathscr{T}_2$ if $\mathscr{T}_1-\mathrm{cl}\ G \subseteq \mathscr{T}_2-\mathrm{cl}\ G$ for all subsets G of X.

It can be shown that $\mathscr{T}_1$ is coupled to $\mathscr{T}_2$ iff for each point x, the $\mathscr{T}_1$-closure of any $\mathscr{T}_2$-neighbourhood of x is a $\mathscr{T}_1$-neighbourhood of x. It is obvious that $\mathscr{T}_1$ is coupled to $\mathscr{T}_2$ if $\mathscr{T}_1 \supseteq \mathscr{T}_2$. Moreover, if $\mathscr{T}_1$ is coupled to $\mathscr{T}_2$, then it is coupled to any topology which is smaller than $\mathscr{T}_2$.

Consistency of $\mathscr{T}_1$ and $\mathscr{T}_2$ is defined just as the pairwise Hausdorffness defined above.

It should be noted that regularity with respect to some other topology is not the same as the property of being coupled to that topology, although the definitions seem to be very much similar. If $\mathscr{T}_1$ is regular with respect to $\mathscr{T}_2$ and $\mathscr{T}_2$ is coupled to $\mathscr{T}_1$, then $\mathscr{T}_1 \subseteq \mathscr{T}_2$. Thus if $\mathscr{T}_1$ and $\mathscr{T}_2$ are p-regular and coupled to each other, then $\mathscr{T}_1 = \mathscr{T}_2$ and the resulting single topology is regular.

When $\mathscr{T}_1$ is coupled to $\mathscr{T}_2$ then consistency implies that $\mathscr{T}_1$ is a Hausdorff topology; the reverse implication is obtained if $\mathscr{T}_1 \subseteq \mathscr{T}_2$. When $(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-Hausdorff, then $\mathscr{T}_1$ and $\mathscr{T}_2$ are T_1 topologies.

It was proved by Weston that if $\mathscr{T}_1$ and $\mathscr{T}_2$ are consistent, that is, if $(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-Hausdorff, then every $\mathscr{T}_1$-compact set is $\mathscr{T}_2$-closed and every $\mathscr{T}_2$-compact set is $\mathscr{T}_1$-closed.

We now state some basic generalizations obtained by Kelly. The analogue of Urysohn's Lemma can be stated as follows:

$(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-*normal iff for every pair of disjoint sets F_1 and F_2 such that F_1 is $\mathscr{T}_1$-closed, F_2 is $\mathscr{T}_2$-closed, there exists a function g on X into $[0, 1]$ such that, $g(F_1) = \{1\}$, $g(F_2) = \{0\}$, g is $\mathscr{T}_1$-upper-semi-continuous ($\mathscr{T}_1$-*u.s.c.*) and $\mathscr{T}_2$-lower semi-continuous ($\mathscr{T}_2$-*l.s.c.*).*

Kelly proved that every p-regular space satisfying the second axiom of countability is p-normal.

The generalization of Urysohn's metrization theorem is obtained as below:

Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be p-*regular satisfying the second axiom of countability. Then X is quasi-pseudo-metrizable. If in addition, X is* p-*Hausdorff, it is quasi-metrizable.*

Kelly proved also the following generalization of Tietze's extension theorem:

If $(X, \mathscr{T}_1, \mathscr{T}_2)$ be p-*normal and A be a subset of X which is both $\mathscr{T}_1$- and $\mathscr{T}_2$-closed and if f be a real-valued function defined on A which is $\mathscr{T}_1$-*u.s.c.* and $\mathscr{T}_2$-*l.s.c.*, then there exists an extension g of f to the whole space such that f is $\mathscr{T}_1$-*u.s.c.* and $\mathscr{T}_2$-*l.s.c.**

E. P. Lane [38], however gave an example to show that the above result obtained by Kelly is incorrect. He proved that the theorem remains true if "real-valued" be replaced by "bounded real-valued". To prove his result, he incidentally obtained the following characterization of p-normal spaces analogous to the characterization of normal spaces proved by M. Katětov [33] and H. Tong [82]:

$(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-*normal iff for every pair of functions f and g defined on X such that f is $\mathscr{T}_1$-*l.s.c. *and g is $\mathscr{T}_2$-*u.s.c. *and $g \leqq f$, there exists a $\mathscr{T}_1$-*l.s.c. *and $\mathscr{T}_2$-*u.s.c. *function h on X such that $g \leqq h \leqq f$.*

Complete-regularity in bitopological spaces was also introduced by Lane. He introduced the following definitions:

*Let A and B be any two subsets of X. Then A is said to be $\mathscr{T}_1$-completely-separated with respect to $\mathscr{T}_2$ from B in case there is a $\mathscr{T}_1$-*l.s.c. *and $\mathscr{T}_2$-*u.s.c. *function f on X such that $f(A) = \{0\}, f(B) = \{1\}$ and $0 \leqq f \leqq 1$. A is $\mathscr{T}_1$-completely-separated with respect to $\mathscr{T}_2$ from B iff B is $\mathscr{T}_2$-completely-separated with respect to $\mathscr{T}_1$ from A.*

p-completely-regular. *$\mathscr{T}_1$ is said to be completely-regular with respect to $\mathscr{T}_2$ in case every $\mathscr{T}_1$-closed subset F of X is $\mathscr{T}_1$-completely-separated with respect to $\mathscr{T}_2$ from each point in $X \sim F$. The space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be* p-*completely-regular if $\mathscr{T}_1$ is completely-regular with respect to $\mathscr{T}_2$ and $\mathscr{T}_2$ is completely-regular with respect to $\mathscr{T}_1$.*

If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-*normal and $\mathscr{T}_1$ and $\mathscr{T}_2$ are T_1-topologies, then $(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-completely-regular. Also, if $\mathscr{T}_1$ is completely-regular with respect to $\mathscr{T}_2$, then $\mathscr{T}_1$ is regular with respect to $\mathscr{T}_2$.*

*If f is a real-valued function on X which is $\mathscr{T}_1$-*l.s.c. *and $\mathscr{T}_2$-*u.s.c. *then $\{x: f(x) \leqq 0\}$ is a $\mathscr{T}_1$-zero set with respect to $\mathscr{T}_2$. A $\mathscr{T}_1$-zero set with respect to $\mathscr{T}_2$ will be called a $\mathscr{T}_1$-zero set and a $\mathscr{T}_2$-zero set with respect to $\mathscr{T}_1$ will be called $\mathscr{T}_2$-zero set. Obviously, every $\mathscr{T}_1$-zero set is $\mathscr{T}_1$-closed and every $\mathscr{T}_2$-zero set is $\mathscr{T}_2$-closed.*

Thus it follows that the space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is p-*completely regular iff the $\mathscr{T}_1$-zero sets form a base for the $\mathscr{T}_2$-closed sets and the $\mathscr{T}_2$-zero sets form a base for the $\mathscr{T}_1$-closed sets.*

Lane proved also, the following result:

If X is p-*normal and if a subset A of X is a $\mathscr{T}_1$-zero set and a $\mathscr{T}_2$-zero set, then every $\mathscr{T}_1$-*u.s.c. *and $\mathscr{T}_2$-*l.s.c. *function on A has an extension to the whole space X.*

The concept of pairwise perfect-normality was introduced independently by E. P. Lane and C. W. Patty. It is defined as below:

A subset A of Z is called $\mathscr{T}_1$-G_δ if A is a countable intersection of $\mathscr{T}_1$-open sets. $\mathscr{T}_2$-G_δ sets are defined similarly.

p-perfectly-normal. *The space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is called pairwise perfectly normal in case X is pairwise normal and every $\mathscr{T}_1$-closed (resp. $\mathscr{T}_2$-closed) subset of X is a $\mathscr{T}_2$-G_δ (resp. $\mathscr{T}_1$-G_δ).*

C. W. Patty [49] defined pairwise completely-normal spaces as below:

p-completely-normal. *$(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be pairwise completely-normal if for every pair of sets A and B such that $\mathscr{T}_1-\mathrm{cl}(A) \cap B = \emptyset$ and $A \cap \mathscr{T}_2-\mathrm{cl}(B) = \emptyset$, there exists a $\mathscr{T}_2$-open set U and a $\mathscr{T}_1$-open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.*

Patty proved that every pairwise perfectly-normal space is pairwise completely-normal.

If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise normal then X is pairwise perfectly-normal iff every $\mathscr{T}_1$-closed subset of X is a $\mathscr{T}_1$-zero set and every $\mathscr{T}_2$-closed subset is a $\mathscr{T}_2$-zero set.

As already mentioned, Kelly generalized Urysohn's metrization theorem for bitopological spaces. Generalizing the Nagata-Smirnov-metrization theorem Lane obtains a sufficient condition for a bitopological space to be quasimetrizable. It is not known whether the condition is necessary. However, this is stronger than Kelly's result. The theorem reads as below:

Suppose that the space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise regular. If there is a sequence $\gamma_n = \{\Gamma_{n\alpha}\}_\alpha$ $(n = 1, 2, \ldots)$ of $\mathscr{T}_1$- and $\mathscr{T}_2$-locally finite $\mathscr{T}_2$-open families such that $\gamma = \bigcup_{n=1}^{\infty} \gamma_n$ is a basis for $\mathscr{T}_2$, and there is a sequence $\tau_n = \{\tau_{n\beta}\}_\beta$ $(n = 1, 2, \ldots)$ of $\mathscr{T}_1$- and $\mathscr{T}_2$-locally finite $\mathscr{T}_1$-open families such that $\tau = \bigcup_{n=1}^{\infty} \tau_n$ is a basis for $\mathscr{T}_1$, then $(X, \mathscr{T}_1, \mathscr{T}_2)$ is quasi-pseudo-metrizable. If, in addition, X is p-Hausdorff, then X is quasi-metrizable.

Lane investigated also, the relationship between quasi-uniform spaces and the associated bitopological spaces. Let $(X, \mathscr{U})$ be a quasi-uniform space. If $x \in X$ and if $V \in \mathscr{U}$, let $V(x) = \{y \in X: (x, y) \in V\}$. There is a unique topology $T(\mathscr{U})$ on X such that $\{V(x): V \in \mathscr{U}\}$ is the filter of all neighbourhoods of x. It is easy to verify that $\mathscr{U}^{-1} = \{V^{-1}: V \in \mathscr{U}\}$ is also a quasi-uniformity on X. Then in the same manner, $\mathscr{U}^{-1}$ determines a unique topology $T(\mathscr{U}^{-1})$ on X. Thus with any quasi-uniform space $(X, \mathscr{U})$ there is associated a bitopological space $(X, T(\mathscr{U}), T(\mathscr{U}^{-1}))$ ($\mathscr{U}$ and $\mathscr{U}^{-1}$ are called conjugate quasi-uniformities and $T(\mathscr{U})$, $T(\mathscr{U}^{-1})$ are called conjugate topologies. $(X, \mathscr{U}, \mathscr{U}^{-1})$ is sometimes referred to as a bi-quasi-uniform space (bq.u.s.)). By considering $(X, \mathscr{U}, \mathscr{U}^{-1})$ we regain some of the symmetry lost in a quasi-uniformity.

A space $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be quasi-uniformizable (named as pairwise uniform by P. Fletcher* [20]*) in case there exists a quasi-uniformity* $\mathscr{U}$ *on* X *such that* $T(\mathscr{U}) = \mathscr{T}_1$ *and* $T(\mathscr{U}^{-1}) = \mathscr{T}_2$. (If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is an arbitrary bitopological space, then according to Pervin's result, there exists a quasi-uniformity $\mathscr{U}$ on X such that $T(\mathscr{U}) = \mathscr{T}_1$; however it is clear that, in general, $T(\mathscr{U}^{-1})$ need have no relationship to $\mathscr{T}_2$).

Lane proves in his paper, the following:

A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is quasi-uniformizable iff it is pairwise completely-regular.*

The above result was proved independently by P. Fletcher [20].

Note that if $(X, \mathscr{T}_1, \mathscr{T}_2)$ be any bitopological space then it is possible to construct a quasi-uniformity $\mathscr{U}$ on X such that $T(\mathscr{U}) = \mathscr{T}_1$. However, an example can be given to show that even if $(X, \mathscr{T}_1, \mathscr{T}_2)$ be p-completely-regular, $T(\mathscr{U}^{-1})$ need not coincide with $\mathscr{T}_2$.

Some more separation axioms, namely, p-T_0, p-T_1 etc. were introduced by M. G. Murdeshwar and S. A. Naimpally in their monograph: Quasi-uniform topological spaces.

p-T_0. *For every pair of distinct points, there exists a* $\mathscr{T}_1$*- or a* $\mathscr{T}_2$*-neighbourhood of one point not containing the other.*

p-T_1. *For every pair of distinct points* x, y, *there exists a* $\mathscr{T}_1$*- or a* $\mathscr{T}_2$*-neighbourhood of* x *not containing* y.

Following are the results which appear in the monograph by Murdeshwar and Naimpally for pairwise Hausdorff spaces.

1. $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is* p-T_2 *iff* Δ *is closed in* $(X \times X, \mathscr{T})$ *where* $\mathscr{T}$ *is* $\mathscr{T}_1 \times \mathscr{T}_2$ *or* $\mathscr{T}_2 \times \mathscr{T}_1$.

2. X *is* p-T_2 *iff for each filter* F *on* X, $F \overset{\mathscr{T}_i}{\rightarrow} x$ *and* $F \overset{\mathscr{T}_j}{\rightarrow} y, i \neq j$, *implies* $x = y$.

3. *A* bq.u.s. $(X, \mathscr{A}_1, \mathscr{A}_2)$ *is* p-T_2 *iff* $\Delta = \bigcap V^{-1} \circ U$ *as* V *runs through* $\mathscr{A}_i$ *and* U *runs through* $\mathscr{A}_j$, $i \neq j$.

p-R_0: *For every* G *in* $\mathscr{T}_i$, $x \in G \Rightarrow \mathscr{T}_j - \text{cl}\,\{x\} \subseteq G$ $(i \neq j)$.

p-R_1: *For* $x, y \in X$ *and* $i \neq j$, $\mathscr{T}_i - \text{cl}\,\{x\} \neq \mathscr{T}_j - \text{cl}\,\{y\} \Rightarrow x$ *has a* $\mathscr{T}_j$*-neighbourhood and* y *has a* $\mathscr{T}_i$*-neighbourhood which are disjoint.*

1. *A* bq.u.s. $(X, \mathscr{A}_1, \mathscr{A}_2)$ *is pairwise* -R_0 *iff* $\bigcap_{\mathscr{A}_i} U = \bigcap_{\mathscr{A}_j} U^{-1}$ $(i \neq j)$.
   Pairwise -$R_1 \Rightarrow$ *Pairwise* -R_0.
2. *For a space* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *the following are equivalent*:
   (a) $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is pairwise* -R_1.
   (b) $\tilde{\Delta} = \{(x, y)\colon \mathscr{T}_1 - \text{cl}\,\{x\} = \mathscr{T}_2 - \text{cl}\,\{y\}\} = \bar{\Delta}$.

(c) $\tilde{\Delta}$ *is closed in* $\mathscr{T}_1 \times \mathscr{T}_2$.

3. *A* bq.u.s *is pairwise* -R_1 *iff* $\tilde{\Delta} = \bigcap V^{-1} \circ U$ *as* V *runs through* $\mathscr{A}_i$ *and* U *through* $\mathscr{A}_j$, $i \neq j$.

4. *A* bq.u.s. *is pairwise* -T_2 *iff it is pairwise* -T_1 *and pairwise* -R_1.

Following result are also proved:

1. *Every compact* $((X, \mathscr{T}_1), (X, \mathscr{T}_2)$ *both compact*), p-R_1 *space is pairwise-normal.*

2. *Every Lindelöf, pairwise-regular space is pairwise-normal.*

Recently, Y. W. Kim [36] has introduced the notion of pairwise-compact spaces.

p-**compact.** $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-*compact if every proper,* $\mathscr{T}_2$*-closed set is* $\mathscr{T}_1$*-compact and every proper* $\mathscr{T}_1$*-closed set is* $\mathscr{T}_2$*-compact.*

With this definition, Kim proves among other results, the following:

1. *Every pairwise-compact, pairwise-Hausdorff space is pairwise-regular.*

2. *Every pairwise-compact, pairwise-regular space is pairwise-normal.*

3. *Every pairwise-compact, pairwise-Hausdorff space is pairwise-uniform and hence also pairwise-completely-regular.*

Pairwise compactness has also been defined and studied by P. Fletcher, H. B. Hoyle III and C. W. Patty [21]. However, Kim's definition is more general.

We now proceed to give an account of our own work on separation axioms in bitopological spaces.

About p-T_0, p-T_1 and p-T_2 spaces, following results have been proved:

1. *Every pairwise*-T_0, *pairwise-regular space is pairwise* -T_1 *and hence pairwise* -T_3.

2. *The following are equivalent:*

   (i) $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is* p-T_1.

   (ii) $\{x\} = \mathscr{T}_1 - \text{cl}\{x\} \cap \mathscr{T}_2 - \text{cl}\{x\}$ *for each* $x \in X$.

   (iii) $\mathscr{T}_1 - d\{x\} \cap \mathscr{T}_2 - d\{x\} = \emptyset$ *for each* $x \in X$. (*Here* $\mathscr{T}_i - d\{x\}$ *denotes derived set of* $\{x\}$ *relative to the topology* $\mathscr{T}_i$.)

   (iv) *The intersection of all* $\mathscr{T}_1$*-neighbourhoods and all* $\mathscr{T}_2$*-neighbourhoods of* x *is equal to* $\{x\}$.

3. $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is* p-T_2 *if and only if for each point* $x \in X$, $\{x\}$ = *the intersection of* $\mathscr{T}_1$*-closures of* $\mathscr{T}_2$*-neighbourhoods of* x = *the intersection of* $\mathscr{T}_2$*-closures of* $\mathscr{T}_1$*-neighbourhoods of* x.

4. *Every subspace of a* p-T_1 (p-T_2) *space is* p-T_1 (p-T_2) *and the product of*

a family of p-T_1 (p-T_2) *spaces is* p-T_1 (p-T_2) (*by the product of* $\{(X_\alpha, \mathcal{T}_\alpha, \mathcal{T}^*_\alpha)$: $\alpha \in \Lambda\}$ *is meant the set* $\prod_{\alpha \in \Lambda} X_\alpha$ *equipped with* $\prod_{\alpha \in \Lambda} \mathcal{T}_\alpha$ *and* $\prod_{\alpha \in \Lambda} \mathcal{T}^*_\alpha$).

The following are some of the results obtained concerning p-regular, p-normal and p-completely-regular spaces:

1. *Result* (4) *above remains true if* "p-T_1" *be replaced by* "p-*regular*" *or* "p-*completely-regular*".

2. *Let* X *be a non-empty set and let* $\{\mathcal{T}_\alpha: \alpha \in \Lambda\}$, $\{\mathcal{T}^*_\alpha: \alpha \in \Lambda\}$ *be two families of topologies for* X. *Let* $\mathcal{T} = \text{lub}\{\mathcal{T}_\alpha: \alpha \in \Lambda\}$ *and* $\mathcal{T}^* = \text{lub}\{\mathcal{T}^*_\alpha: \alpha \in \Lambda\}$. *If* $(X, \mathcal{T}_\alpha, \mathcal{T}^*_\alpha)$ *is* p-*regular* (p-*completely-regular*, p-T_1, p-T_2) *for each* $\alpha \in \Lambda$, *then so is* $(X, \mathcal{T}, \mathcal{T}^*)$.

3. *For any bitopological space* $(X, \mathcal{T}_1, \mathcal{T}_2)$, *there exist unique topologies* $\mathcal{T}^*_1, \mathcal{T}^*_2$ *for* X *such that* $\mathcal{T}^*_1 \subseteq \mathcal{T}_1$, $\mathcal{T}^*_2 \subseteq \mathcal{T}_2$ $(X, \mathcal{T}^*_1, \mathcal{T}^*_2)$ *is* p-*regular* (p-*completely-regular*) *and if* Y *be any* p-*regular* (p-*completely-regular*) *space, then the pairwise continuous maps*, $(X, \mathcal{T}_1, \mathcal{T}_2) \to Y$ *are precisely the pairwise continuous maps* $(X, \mathcal{T}^*_1, \mathcal{T}^*_2) \to Y$ ($f: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ *is said to be pairwise continuous if* $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ *and* $f: (X, \mathcal{T}_2) \to (Y, \mathcal{U}_2)$ *are both continuous*). *Further*, $(\mathcal{T}^*_1, \mathcal{T}^*_2)$ *is the least upper bound of all* p-*regular* (p-*completely-regular*) *topologies* $(\mathcal{U}_1, \mathcal{U}_2)$ *for* X *in the sense that if* $\mathcal{U}_1 \subseteq \mathcal{T}_1$, $\mathcal{U}_2 \subseteq \mathcal{T}_2$ *then* $\mathcal{U}_1 \subseteq \mathcal{T}^*_1$, $\mathcal{U}_2 \subseteq \mathcal{T}^*_2$.

4. *Let* f *be a pairwise-closed* (f: $(X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ *is said to be pairwise closed if* $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ *and* $f: (X, \mathcal{T}_2) \to (X, \mathcal{U}_2)$ *are both closed*) *and pairwise continuous mapping of* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *onto* $(Y, \mathcal{T}^*_1, \mathcal{T}^*_2)$. *If* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *is p-regular and for each* $y \in Y$, $f^{-1}(y)$ *is* $\mathcal{T}_1$-*compact as well as* $\mathcal{T}_2$-*compact, then* $(Y, \mathcal{T}^*_1, \mathcal{T}^*_2)$ *is* p-*regular*.

5. *Every* p-*closed*, p-*continuous image of a* p-*normal space is* p-*normal*.

N. Levine [40] introduced the notion of simple extensions in topological spaces as below:

Let $(X, \mathcal{T})$ *be a topological space and let* $A \notin \mathcal{T}$. *Then the family* $\mathcal{T}^*(A) = \{G \cup (G^* \cap A): G, G^* \in \mathcal{T}\}$ *is a topology for* X, *called a simple extension of* $\mathcal{T}$.

We have then proved the following results:

1. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be a bitopological space and let* $A \notin \mathcal{T}_1$, $A \notin \mathcal{T}_2$. *Let* $\mathcal{T}_1(A)$ *be a simple extension of* $\mathcal{T}_1$ *and* $\mathcal{T}_2(A)$ *be a simple extension of* $\mathcal{T}_2$. *Then if* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *is* T_0, T_1 *or* T_2, *so is* $(X, \mathcal{T}_1(A), \mathcal{T}_2(A))$.

2. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be* p-*regular. If* A *is* $\mathcal{T}_1$-*closed as well as* $\mathcal{T}_2$-*closed, then* $(X, \mathcal{T}_1(A), \mathcal{T}_2(A))$ *is also* p-*regular*.

3. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be* p-*normal. Let* A *be* $\mathcal{T}_1$-*closed as well as* $\mathcal{T}_2$-*closed. Then*, $(X, \mathcal{T}_1(A), \mathcal{T}_2(A))$ *is* p-*normal if and only if* $(X \sim A, \mathcal{T}_1 \cap X \sim A, \mathcal{T}_2 \cap \cap X \sim A)$ *is* p-*normal*.

We have introduced and studied some new separation axioms in bitopological spaces.

1. p-T_D. $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-T_D *if* $\mathscr{T}_1 - d\{x\} \cap \mathscr{T}_2 - d\{x\}$ *is* $\mathscr{T}_1$*-closed as well as* $\mathscr{T}_2$*-closed for each* $x \in X$.

Obviously, p-$T_1 \Rightarrow$ p-$T_D \Rightarrow$ p-T_0.

There exist examples to show that the reverse implications do not hold in general.

2. p-kc: $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-kc *if every* $\mathscr{T}_1$*-compact set is* $\mathscr{T}_2$*-closed and every* $\mathscr{T}_2$*-compact set is* $\mathscr{T}_1$*-closed.*

3. p-us: $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-us *if for every sequence* $\langle x_n \rangle$ *in* X *such that* $x_n \overset{\mathscr{T}_1}{\to} x$ *(that is* $\langle x_n \rangle$ *converges to* x *relative to* $\mathscr{T}_1$*) and* $x_n \overset{\mathscr{T}_2}{\to} y$, *we must have* $x = y$.

It is clear that, p-$T_2 \Rightarrow$ p-kc $\Rightarrow$ p-us $\Rightarrow$ p-T_1.

However, in a bi-first axiom space $(X, \mathscr{T}_1, \mathscr{T}_2)$ *(that is,* $(X, \mathscr{T}_1)$, $(X, \mathscr{T}_2)$ *are both first axiom), we have,* p-$T_2 \Leftrightarrow$ p-kc $\Leftrightarrow$ p-us.

Following are some more results that we have obtained:

1. *Let* f *be* p-*closed,* p-*continuous mapping of a* p-kc *space* X *onto a space* Y *such that* $f^{-1}(y)$ *is both,* $\mathscr{T}_1$*-compact and* $\mathscr{T}_2$*-compact. Then* Y *is* p-kc.

2. *The property of being* p-us *is preserved under one-to-one, onto and p-open maps and it is inversely preserved under one-to-one, onto, p-continuous maps.*

3. *In a* p-us *space, every sequentially* $\mathscr{T}_1$*-compact set is sequentially* $\mathscr{T}_2$*-closed and every sequentially* $\mathscr{T}_2$*-compact set is sequentially* $\mathscr{T}_1$*-closed.*

4. $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is* p-us *if and only if the diagonal* Δ *in* $X \times X$ *is sequentially closed in* $(X \times X, \mathscr{T})$ *where* $\mathscr{T}$ *is either* $\mathscr{T}_1 \times \mathscr{T}_2$ *or* $\mathscr{T}_2 \times \mathscr{T}_1$.

4. p-**Urysohn.** $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-*Urysohn if for any two points* x *and* y *of* X *such that* $x \neq y$, *there exists a* $\mathscr{T}_1$*-open set* U *and a* $\mathscr{T}_2$*-open set* V *such that* $x \in U$, $y \in V$, $\mathscr{T}_2 - \text{cl.}\, U \cap \mathscr{T}_1 - \text{cl.}\, V = \emptyset$.

Obviously, p-$T_3 \Rightarrow$ p-Urysohn $\Rightarrow$ p-T_2.

5. p-**semi-regular.** $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-*semi-regular if for every* $\mathscr{T}_i$*-open set* U *containing a point* x, *there exists a* $\mathscr{T}_j$*-open set* V *such that* $x \in V \subseteq \mathscr{T}_i - \text{int.}\, \mathscr{T}_j - \text{cl.}\, V \subseteq U$, $i, j = 1, 2$, $i \neq j$.

Call a subset A *of* $(X, \mathscr{T}_1, \mathscr{T}_2)$ (i, j)*-regularly closed if* $A = \mathscr{T}_i - \text{cl.}\, \mathscr{T}_j - \text{int.}\, A$, $i, j = 1, 2$, $i \neq j$.

6. p-**almost-regular.** $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is said to be* p-*almost regular if for every* (1,2)*-regularly closed ((2, 1)-regularly closed) set* F *and a point* $x \notin F$, *there*

exist a $\mathscr{T}_2$*-open* ($\mathscr{T}_1$*-open*) *set* V *and a disjoint* $\mathscr{T}_1$*-open* ($\mathscr{T}_2$*-open*) *set* U *such that* $x \in U$, $F \subseteq V$.

We have proved that

p-*almost-regular* + p-*semi-regular* = p-*regular, and* p-T_2 + p-*almost-regular* $\Rightarrow$ $\Rightarrow$ p-*Urysohn.*

Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be any bitopological space and "$\leqq$" be a binary relation in X. We define the concept of pairwise-strong-normality as below:

p-strong-normality. *For any two sets* A_1 *and* A_2 *such that* $A_1 \cap A_2 = \emptyset$, A_1 *is* $\mathscr{T}_1$*-closed,* A_2 *is* $\mathscr{T}_2$*-closed and either* A_1 *is decreasing or* A_2 *is increasing, there exist disjoint monotone sets* U_1 *and* U_2 *such that* $A_1 \subseteq U_1$, $A_2 \subseteq U_2$ *and* U_1 *is decreasing,* U_2 *is increasing,* U_1 *is* $\mathscr{T}_2$*-open,* U_2 *is* $\mathscr{T}_1$*-open.*

The analogue of Urysohn's Lemma for such spaces is obtained as below:

A space $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is pairwise-strongly-normal iff for every pair of disjoint sets* A *and* B *in* X *such that* A *is* $\mathscr{T}_1$*-closed,* B *is* $\mathscr{T}_2$*-closed, either* A *is decreasing or* B *is increasing, there exists a function* $h: X \to [0, 1]$ *such that*

(i) $h(A) = \{0\}$, $h(B) = \{1\}$,

(ii) $h(x) \leqq h(y)$ *for all* $x \leqq y$,

(iii) h *is* $\mathscr{T}_2$-u.s.c. *and* $\mathscr{T}_1$-l.s.c.

Note that

(a) *if* "$\leqq$" *is the trivial relation then the above reduces to the result of Kelly obtained for bitopological spaces.*

(b) *If* $\mathscr{T}_1 = \mathscr{T}_2$, *then it reduces to Ward's result for strongly-normal spaces.*

(c) *If* "$\leqq$" *is the trivial relation and* $\mathscr{T}_1 = \mathscr{T}_2$ *then this result is nothing but the well known Urysohn's Lemma for general topological spaces.*

6. Separation Axioms and co-Topologies

The concept of co-topologies was introduced by J. De Groot [17]. In his thesis [76], G. E. Strecker obtains some results concerning separation axioms in relation to co-topologies. The definition of co-topologies used by Strecker is more general than the one used by De Groot in [17]. However, the following is De Groot's original definition.

Let $\mathfrak{B}$ be a base for the space $(X, \mathscr{T})$. Then the *co-topology* of $\mathscr{T}$ with respect to $\mathfrak{B}$ (denoted by $\mathscr{T}_{\mathfrak{B}}$) is the topology generated by the family consisting of the complements of closures of members of $\mathfrak{B}$. The space $(X, \mathscr{T}_{\mathfrak{B}})$ will be called the *co-space with respect to* $\mathfrak{B}$ of the generating space $(X, \mathscr{T})$.

Let $\mathfrak{P}$ be some topological property. A space with some co-topology having property $\mathfrak{P}$ will be called co-$\mathfrak{P}$. If every co-topology has property $\mathfrak{P}$, then the generating space will be called totally co-$\mathfrak{P}$.

The following characterizations of T_2 and $T_{2\frac{1}{2}}$ spaces have been obtained:

1. *A space is Hausdorff iff it is* co-*Hausdorff.*
2. *A space is Urysohn iff it is* co-*Urysohn.*

Strecker proves also the following:

1. *Every* co-T_1 (co-T_0) *space is* T_1 (T_0) *but not conversely.*
2. *A* T_1 *space is not necessarily* co-T_0.
3. *A space is* T_1 *iff it is a co-space of a discrete space.*

He considers the following definitions:

A space is said to have property A($\mathfrak{B}$) *provided that* $\mathfrak{B}$ *is a base for the space and for every pair of distinct elements x and y of the space, there exists some* $U \in \mathfrak{B}$ *such that* $x \in \overline{U}$ *and* $y \notin \overline{U}$.

A space is said to have property A *iff it has* A($\mathfrak{B}$) *for some base* $\mathfrak{B}$.

A space is said to have property T(A) *iff it has* A($\mathfrak{B}$) *for every base* $\mathfrak{B}$.

A space has A *iff it has* A($\mathcal{T}$).

Also, $T_2 \Rightarrow T(A) \Rightarrow A \Rightarrow T_1$.

Strecker shows that all reverse implications are false. He also shows that A $\Leftrightarrow$ co-T_1, T(A) $\Leftrightarrow$ totally co-T_1.

Thus, A ($\Leftrightarrow$ co-T_1) and T(A) ($\Leftrightarrow$ totally co-T_1) may be looked upon as distinct separation axioms between T_1 and T_2.

The following results have also been proved:

Every semi-regular (resp. regular) space is co-*semi-regular (resp.* co-*regular) but not conversely.*

It is also proved that

Totally co-*Hausdorff* $\Leftrightarrow$ *totally* co-*semi*-T_3 *and that totally* co-T_3 $\Leftrightarrow$ *totally* co-*Tychonoff* $\Leftrightarrow$ *totally* co-T_4 $\Leftrightarrow$ *totally* co-*Urysohn.*

As regards products it is proved that the product of a collection of spaces is totally co-T_3, totally co-T_4, totally co-T_2, totally co-Urysohn, totally co-Tychonoff iff each co-ordinate space has the same property.

Strecker poses the problem whether the statement "the product of totally co-$\mathfrak{P}$ spaces is totally co-$\mathfrak{P}$" is true for every topological property $\mathfrak{P}$ which is preserved under the formation of products.

7. Recent Results Concerning Standard Separation Axioms

1. W. J. Pervin and H. J. Biesterfeldt, Jr. [50] have given a characterization of regularity using the concept of iterate net.[1])

Iterate net. *A net* $\{x_m; m \in D\}$ *is called an iterate net in a space* X *if for each* $m \in D$, *there is a net* $\{x_d^m: d \in D_m\}$ *converging to* x_m.

The net $\{x^m_{\beta(m)}: \langle m, \beta\rangle \in D \times \prod_{m\in D} D_m\}$, where the product set is directed by the product order, is called the composite net of the system of nets.

Pervin and Biesterfeldt have proved that a space is regular iff every iterate net converges to the limit of the composite net whenever that limit exists.

2. Recently, J. P. Thomas [79] has obtained the following result concerning regular topologies on a set X:

If $\mathcal{T}$ *is a topology on* X, *then there is a unique regular topology* $\mathcal{T}_*$, *coarser than* $\mathcal{T}$, *such that if* Y *is any regular space, the continuous maps* $(X, \mathcal{T}) \to Y$ *are precisely the continuous maps* $(X, \mathcal{T}_*) \to Y$. *Further,* $\mathcal{T}_*$ *is the least upper bound of the regular topologies coarser than* $\mathcal{T}$.

It can be proved that the above result of Thomas remains true if "regular" be replaced by "completely regular".

3. Recently, J. D. Groot and J. M. Aarts [18] have obtained two characterizations of completely regular spaces. For this, they introduce the notion of screening as below:

Two subsets A *and* B *of a space* X *are said to be screened by the pair* (C, D) *if* $C \cup D = X$, $A \cap D = \emptyset$ *and* $C \cap B = \emptyset$ *(hence,* $A \subseteq C$ *and* $B \subseteq D$*).*

Groot and Aarts then prove the following two results.

1. *A* T_1 *space* X *is completely regular if and only if there is a base* $\mathfrak{B}$ *for the closed subsets of* X *such that if* $B \in \mathfrak{B}$ *and* $x \notin B$, *then* $\{x\}$ *and* B *are screened by a pair from* $\mathfrak{B}$ *and every pair of disjoint members of* $\mathfrak{B}$ *are screened by a pair from* $\mathfrak{B}$.

2. *A* T_1 *space* X *is completely regular if and only if there is a subbase* $\mathfrak{S}$ *for the closed subsets of* X *such that if* $S \in \mathfrak{S}$ *and* $x \notin S$, *then* $\{x\}$ *and* S *are screened by a finite subcollection of* $\mathfrak{S}$ *and every pair of disjoint members of* $\mathfrak{S}$ *are screened by a finite subcollection of* $\mathfrak{S}$.

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[1]) Editor's note: see also G. Birkhoff, Ann. of Math. *38* (1937), 39—56, Theorem 7a; G. Grimeisen, Math. Annalen *144* (1961), 386—417; I. Fleischer, Col. Math. *15* (1966), 235—241.

Result (1) above was proved by O. Frink [24] with the additional assumption that all finite unions and intersections of members of $\mathfrak{B}$ belong to $\mathfrak{B}$. This result has also been proved independently by E. F. Steiner [71].

4. M. G. Murdeshwar and S. A. Naimpally [45] have considered separation axioms in quasi-uniform spaces and have obtained characterisations of some of the separation axioms in terms of quasi-uniformities without explicit reference to the induced topology. For T_0, T_1, T_2 the following characterisations have been obtained:

T_0: $\bigcap_{U\in\mathcal{U}} U$ *is anti-symmetric.*

T_1: $\Delta = \bigcap_{U\in\mathcal{U}} U$.

T_2: $\Delta = \bigcap_{U\in\mathcal{U}} U^{-1} \circ U$.

The following two characterisations for regularity have also been obtained:

(i) *There exists a quasi-uniformity $\mathcal{U}$ compatible with $\mathcal{T}$ such that for each $x \in X$ and $U \in \mathcal{U}$, there is a symmetric $V \in \mathcal{U}$ with $(V \circ V)[x] \subseteq U[x]$.*

(ii) *There exists a quasi-uniformity $\mathcal{U}$ compatible with $\mathcal{T}$ such that for each $x \in X$ and $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with*

$$(V^{-1} \circ V)[x] \subseteq U[x].$$

In the same paper, Murdeshwar and Naimpally prove also the following two results:

(i) *If a space $(X, \mathcal{T})$ is regular, then there exists a compatible quasi-uniformity $\mathcal{U}$ such that*

$$\bigcap_{U\in\mathcal{U}} U^{-1} \circ U = \bigcap_{U\in\mathcal{U}} U = \bigcap_{U\in\mathcal{U}} U \circ U^{-1}.$$

(ii) *Let $(X, \mathcal{U})$ be a quasi-uniform space such that for each subset A of X,*

$$\bar{A} = \bigcap_{U\in\mathcal{U}} U[A].$$

Then $(X, \mathcal{U})$ is regular.

In their monograph, Murdeshwar and Naimpally prove a result much better than (ii) above:

A space $(X, \mathcal{T})$ is completely-regular iff there exists a compatible quasi-uniformity $\mathcal{A}$ such that for every $A \subseteq X$, $\bar{A} = \bigcap_{U\in\mathcal{U}} U[A]$.

5. P. H. Doyle and J. G. Hocking [19] initiated a study of invertible spaces which are defined as below:

A space X is said to be invertible with respect to an open subset U of X if there exists a homeomorphism h of X onto X such that $h(X \sim U) \subseteq U$.

A space which is invertible with respect to every open set is called invertible.

Doyle and Hocking proved that if a space X is invertible with respect to an open set U of X which is T_i $(i = 0, 1, 2)$, then X is a T_i $(i = 0, 1, 2)$ space.

Similar results were obtained by N. Levine [39] for regular and normal spaces.

Y. M. Wong [89] obtained several results for separation axioms in invertible spaces and proved that the results of Doyle, Hocking and Levine follow from his results as particular cases. Some of the results of Doyle and Hocking mentioned above were also obtained by David Ryeburn [56] by using more general methods. He proved the following:

If X is a space invertible with respect to an open subset U of X and $\bar{U}$ is regular (T_3, *normal or* T_4), *then X is regular* (T_3, *normal or* T_4).

D. X. Hong [31] introduces the concept of generalized invertible spaces as follows:

A space X is said to be generalized invertible if there is a proper, open subset U of X and a homeomorphism h of X onto X such that for each $x \in X$, $h^n(x) \in U$ for some integer $n(x)$. The pair (U, h) is called an inverting pair for X.

Every Euclidean space E^n is a generalized invertible space, although E^n is not an invertible space.

Hong proves that if (U, h) is an inverting pair for a generalized invertible space X and U is T_0 (or T_1), then X is T_0 (or T_1). He also shows that the corresponding results for Hausdorff and regular spaces are not necessarily true. However, he proves that if $U \subseteq A$ where A is closed, then A is Hausdorff (or regular) would imply that X is Hausdorff (or regular).

6. N. Levine [40] obtains sufficient conditions for $(X, \mathscr{T})$ to inherit properties of regularity, normality and complete-regularity etc. from $(X, \mathscr{T}^*)$ where $\mathscr{T}^*$ is a simple extension of $\mathscr{T}$.

He proves that if $(X, \mathscr{T})$ is a regular (T_3, completely-regular or Tychonoff) space, then $(X, \mathscr{T}(A))$ is also regular (T_3, completely-regular or Tychonoff) provided $X \sim A \in \mathscr{T}$. For normality, the following result is proved: Let $(X, \mathscr{T})$ be normal and let $X \sim A \in \mathscr{T}$. Then $(X, \mathscr{T}(A))$ is normal iff $(X \sim A, \mathscr{T} \cap X \sim A)$ is normal.

Recently, C. J. R. Borges [11] has studied simple extensions of topologies in detail and has obtained necessary and sufficient conditions for $(X, \mathscr{T}(A))$ to inherit several properties from $(X, \mathscr{T})$ thus improving almost all results of Levine. In particular, the results of Levine concerning regularity, complete-regularity and normality mentioned above have been improved as below:

1. *If $(X, \mathscr{T})$ is regular, then $(X, \mathscr{T}(A))$ is regular iff $\bar{A} \sim A$ is a $\mathscr{T}$-closed subset of X (that is, $A \cap$ boundary $(A) \subset X \sim (\bar{A} \sim A) \in \mathscr{T}$).*

2. *If* $(X, \mathscr{T})$ *is completely-regular, then* $(X, \mathscr{T}(A))$ *is completely-regular iff it is regular.*

3. *If* $(X, \mathscr{T})$ *is normal, then* $(X, \mathscr{T}(A))$ *is normal iff it is a regular space and* $X \sim A$ *is a normal subspace of* $(X, \mathscr{T})$.

In the same paper, following two results are also obtained:

1. *If* $(X, \mathscr{T})$ *be hereditarily-normal, then* $(X, \mathscr{T}(A))$ *is hereditarily-normal iff it is regular.*

2. *If* $(X, \mathscr{T})$ *be perfectly-normal, then* $(X, \mathscr{T}(A))$ *is perfectly-normal iff it is regular.*

Some results of Levine and Borges have also been proved by D. Ryeburn using more general methods.

7. R. Dacić [14] introduces the notion of Σ-enlargements as follows:

By Σ*-enlargement of a given topology* $\mathscr{T}$ *is meant a topology* $\mathscr{T}_\Sigma$ *a base for which is the following family of subsets of* X:

$\mathscr{T} \cup \{G \cap S\colon G \in \mathscr{T}, S \in \Sigma\}$ *where* Σ *is a ring of subsets of* X *with respect to intersection and symmetric difference operations.*

The way of introducing the Σ-enlargements is justified by the fact that if $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on a set X and $\mathscr{T} \subseteq \mathscr{T}'$ then there exists a ring Σ such that Σ-enlargement $\mathscr{T}_\Sigma$ of $\mathscr{T}$ is just $\mathscr{T}'$.

Dacić obtains two sufficient conditions for $(X, \mathscr{T}_\Sigma)$ to inherit regularity from $(X, \mathscr{T})$ as follows:

1. *If* $(S \cap G)^0 \neq \emptyset$ *for every* $S \in \Sigma$ *and every* $G \in \mathscr{T}$ *for which* $S \cap G$ *is non-empty and* $(X, \mathscr{T})$ *is regular then,* $\mathscr{T}_\Sigma$ *is also regular.*

2. *If* $(X, \mathscr{T})$ *is regular and every member of* Σ *is closed in* $(X, \mathscr{T})$ *then* $(X, \mathscr{T}_\Sigma)$ *is regular.*

8. In another paper [15], R. Dacić defines a choice topology $\mathscr{T}_Z$ for X/R for any space $(X, \mathscr{T})$ and any equivalence relation R in X and examines as to what axioms when possessed by $(X, \mathscr{T})$ are also possessed by $(X/R, \mathscr{T}_Z)$. Dacić introduces the notion of choice topologies as below:

Let $(X, \mathscr{T})$ *be any space. Let* R *be any equivalence relation in* X. *Let* $D = \{D_\alpha\colon \alpha \in \Lambda\}$ *be the quotient set of* X *modulo* R. *Consider any mapping* $\Phi\colon D \to X$ *defined so that* $\Phi(D_\alpha) \in D_\alpha$. *Such a mapping is called a choice function. If* Z *denotes the family of all choice functions, then the choice topology* $\mathscr{T}_Z$ *is the coarest topology on* D *for which all choice functions are continuous.*

Dacić then proves the following result:

If $(X, \mathscr{T})$ *is* T_1, T_2, T_3 *or* T_4, *then so is* $(D, \mathscr{T}_Z)$.

8. Concluding Remarks

1. A lot of work has been done on minimal and maximal separation properties. It will not be possible here to include a complete account of all these. However, the interested reader is referred to papers 4–9, 12, 26–28, 30, 48, 51–54, 57, 59, 62, 70, 72, 73, 75–78, 81, 84, and 85.

2. H. Sharp [61] characterizes each topology on a finite set $S = \{s_1, s_2, \ldots, s_n\}$ with an $n \times n$ zero-one matrix $T = (t_{ij})$ where $t_{ij} = 1$ iff $s_j \in \{\bar{s}_i\}$. Recently, D. A. Bonnett and J. R. Porter [10] have obtained matrix characterisations of many separation axioms in finite spaces, for example, regular, completely-regular, normal, completely-normal, T_γ, R_0, R_1, strong T_0, strong T_D and the six separation axioms between T_0 and T_1 introduced by Aull and Thron.

3. A. Csazar, in his monograph "Fondements de la topologie générale" published in 1960, introduces the class of syntopogenous spaces. These spaces appear as a generalization of topological, proximity and uniform spaces.

Csazar considered T_0, T_1 and T_2 separation axioms in this more general setting. Other separation axioms in syntopogenous spaces, for example, regular, completely-regular, normal, completely-normal, Urysohn and Stone, were considered by J. L. Sieber and W. J. Pervin [63, 64].

The author wishes to acknowledge her sincere thanks to Prof. M. K. Singal for his constant help and encouragement and his infinite patience in guiding this effort.

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