Shashi Prabha Arya

Some recent work on paracompactness

In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971. pp. [33]--48.

Persistent URL: http://dml.cz/dmlcz/700575

SOME RECENT WORK ON PARACOMPACTNESS

SHASHI PRABHA ARYA

New Delhi

In the present survey, we propose to report on some work on paracompactness which is either unpublished or has been published only recently. We classify the work to be presented into three classes relating to (i) paracompactness (ii) countable paracompactness (iii) $\mathfrak{m}$-paracompactness ($\mathfrak{m}$ being any infinite cardinal number). Besides our own work, the work of the following authors will be considered: James R. Boone, John Greever, Chien Wenjen, E. E. Grace, D. R. Traylor, H. Tamano, W. B. Sconyers, S. Swaminathan, J. E. Vaughan, Y. Katuta, B. H. McCandless, C. E. Aull, Phillip Zenor and John Mack.

1. Paracompactness

1. Sequential spaces which were introduced by S. P. Franklin [13][1]) are proving rather useful and are having their impact on various fields of general topology. Recently, James R. Boone [7] a student of H. Tamano, has obtained characterizations of paracompactness in sequential spaces and also in k-spaces. For this purpose he introduces four weaker forms of paracompactness. To understand these the following basic definitions are needed.

Definition. *A family $\mathscr{F}$ is said to be compact-finite (resp. cs-finite) if every compact set (resp. every closure of a convergent sequence) intersects finitely many members of $\mathscr{F}$.*

Definition. *A family $\mathscr{F}$ is said to be strongly compact-finite (resp. strongly cs-finite) if the family of closures of members of $\mathscr{F}$ is compact-finite (resp.cs-finite).*

The four weaker forms of paracompactness mentioned earlier, can now be defined as follows:

Definition. *A space X is said to be mesocompact (resp. strongly mesocompact, sequentially mesocompact, strongly sequentially mesocompact) if every open*

[1]) Editor's note: See also G. Birkhoff, On the combination of topologies, Fund. Math. *26* (1936), 156–166.

covering of X has a compact-finite (resp. strongly compact-finite, cs-finite, strongly cs-finite) open refinement.

The following are the obvious implications between these various forms of paracompactness:

$$\begin{array}{ccccc} \text{paracompact} & \Rightarrow & \text{strongly mesocompact} & \Rightarrow & \text{mesocompact} \\ & & \Downarrow & & \Downarrow \\ & & \text{strongly sequentially mesocompact} & \Rightarrow & \text{sequentially mesocompact} \\ & & & & \Downarrow \\ & & & & \text{pointwise paracompact} \end{array}$$

Boone's main results can now be summarized as below:

(1) *In a normal space,*
mesocompact (resp. sequentially mesocompact) $\Leftrightarrow$ *strongly mesocompact (resp. strongly sequentially mesocompact).*

(2) *In a locally compact space,*
paracompact $\Leftrightarrow$ *mesocompact*

(3) *In a first axiom space,*
paracompact $\Leftrightarrow$ *sequentially mesocompact*

(4) *In a k-space,*
paracompact $\Leftrightarrow$ *strongly mesocompact*

(5) *In a sequential space,*
paracompact $\Leftrightarrow$ *strongly sequentially mesocompact.*

For k'-spaces [1] and Fréchet spaces [2] of Arhangel'skii, the following results are obtained:

(1) *In a regular k'-space,*
paracompact $\Leftrightarrow$ *mesocompact.*

(2) *In a regular Fréchet space,*
paracompact $\Leftrightarrow$ *sequentially mesocompact.*

It should be noted that Boone has not given examples to show that all the four forms of paracompactness introduced are distinct. However, he gives an example of a normal mesocompact (and hence a strongly mesocompact) space which is not paracompact and that of a regular developable pointwise paracompact space which is not sequentially mesocompact.

Boone has also obtained a sufficient condition for a space to be paracompact by using the concepts of $\mathscr{K}$-collection and W-weak topology which are defined as follows:

Definition. *A collection* $\mathscr{K} = \{K_\alpha : \alpha \in J\}$ *of subsets of a space X is called a* $\mathscr{K}$*-collection if* $\mathscr{K}$ *is a covering of the space X and if for each closed subset F of X,* $F \cap K_\alpha \in \mathscr{K}$ *for each* α.

Definition. *A space X is said to have W-weak topology with respect to a family* $\mathscr{K}$ *if a set G is open iff* $G \cap H$ *is open in H for each* $H \in \mathscr{K}$.

The result that Boone has proved is the following:

If a regular T_1 *space X has the W-weak topology with respect to a* $\mathscr{K}$*-collection* $\mathscr{K}$ *and if every open covering of X has a* $\mathscr{K}$*-finite closed refinement, then X is paracompact.*

We have been able to improve the above result of Boone in two ways as below:

(1) *If a space X has the W-weak topology with respect to a* $\mathscr{K}$*-collection and if every open covering of X has a* $\mathscr{K}$*-finite closed refinement, then X is paracompact.*

(2) *If a space X has the W-weak topology with respect to a* $\mathscr{K}$*-collection and if every directed open covering of X has a* $\mathscr{K}$*-finite closed refinement, then X is paracompact.*

In [8], Boone has obtained two more characterizations of paracompactness in Fréchet and sequential spaces with the help of a new concept – the concept of property (w) which he defines as follows:

Definition. *A Hausdorff space X is said to have the property* (w) *provided: for each discrete collection* $\{F_\alpha : \alpha \in \Lambda\}$ *of closed sets in X, there exists a cs-finite collection* $\{G_\alpha : \alpha \in \Lambda\}$ *of open sets such that* $F_\alpha \subset G_\alpha$, *for each* $\alpha \in \Lambda$ *and* $G_\alpha \cap F_\beta = \emptyset$ *if* $\alpha \neq \beta$.

Spaces with the property (w) are a simultaneous generalization of sequentially mesocompact and collectionwise normal spaces.

Following are the two results proved by Boone:

(a) *A Fréchet space is paracompact iff it is a regular pointwise paracompact space with property* (w).

(b) *A sequential space is paracompact iff it is a normal pointwise paracompact space with property* (w).

2. Mani Gagrat, a student of S. P. Franklin, has obtained some sufficient conditions for a space to be paracompact, using the notions of natural covers and Σ-spaces. The notion of natural covers is due to R. Brown [10].

Definition. *By a natural cover we mean a function Σ which assigns to each space X a covering Σ_X satisfying the following*:

(i) *If $A \in \Sigma_X$ and A is homeomorphic to a subspace B of a space Y, then $B \in \Sigma_Y$, and*

(ii) *If $f : X \to Y$ is a continuous function and $A \in \Sigma_X$, then there is a $B \in \Sigma_Y$ such that $f(A) \subset B$.*

For any natural cover Σ and any space $(X, \mathscr{T})$,

$$\Sigma_X(\mathscr{T}) = \{A \subset X : A \cap U \text{ is open in } U \text{ for each } U \in \Sigma_X\} .$$

If $\Sigma_X(\mathscr{T}) = \mathscr{T}$, then X is said to be a Σ-space.

Gagrat introduces the notions of Σ-compact and strongly Σ-compact spaces.

Definition. *For any natural cover Σ and any space $(X, \mathscr{T})$ a family $\mathscr{F}$ of subsets of X is said to be Σ-finite if every member of Σ_X intersects at most finitely many members of $\mathscr{F}$. $\mathscr{F}$ is said to be strongly Σ-finite if the family of closures of members of $\mathscr{F}$ is Σ-finite.*

Definition. *A space X is said to be Σ-compact (resp. strongly Σ-compact) if every open covering of X has a Σ-finite (resp. strongly Σ-finite) open refinement.*

With these definitions, she obtains the following results:

(1) *A regular (resp. normal) Σ-space X is paracompact if every open covering of X has a Σ-finite closed (resp. open) refinement.*

(2) *If every $S \in \Sigma_X$ is compact, then paracompactness implies Σ-compactness in any regular space X.*

An example has been constructed to show that paracompactness in a Σ-space need not imply Σ-compactness.

3. J. Greever [18] calls a space *hypo-Lindelöf* if every open covering has a star countable open refinement and proves the following result:

A hypo-Lindelöf space is paracompact if and only if it is countably paracompact.

In [19], J. Greever proves that the above result remains true if "hypo-Lindelöf" be replaced by "screenable" where by a *screenable* space is meant a space every open covering of which has a σ-pairwise disjoint open refinement.

4. As is well-known, a compact space is characterized by any of the following equivalent properties:

(1) Every net has a cluster point.

(2) Every open covering of the space of cardinality $\geqq \aleph_0$ (the cardinality of the set of positive integers) has a subcovering of cardinality $< \aleph_0$.

There arises a question as to how do these properties of compact spaces re-appear in paracompact spaces.

C. Wenjen [39] obtains analogues of these two properties for paracompact spaces.

Definition. *Let $\{x_d : d \in D\}$ be any net. The family of the cardinal numbers of all cofinal subsets of D contains a smallest number which is called the least cardinal number of the net $\{x_d : d \in D\}$.*

Wenjen then proves the following result:

If X is a uniform space with the family $\mathscr{V} = \{V_d : d \in D\}$ of all neighbourhoods of the diagonal as a uniformity and $\mathfrak{m}$ is the least cardinal number of the net $\{V_d : d \in D\}$, then the following are equivalent;

(i) *X is paracompact.*

(ii) *Each net in X with the least cardinal number $\geqq \mathfrak{m}$ has a cluster point.*

(iii) *There is a subcover of cardinality $< \mathfrak{m}$ of each open covering of X of cardinality $\geqq \mathfrak{m}$.*

It is well-known that the product of two paracompact spaces may fail to be paracompact. Wenjen has proved that the product of a paracompact space with a locally compact paracompact space is paracompact.

5. E. E. Grace [16] introduces the notion of peripheral paracompactness.

Definition. *A space X is said to be peripherally paracompact in the strong sense (resp. in the weak sense) if, for each frontier set (i.e., each nowhere dense closed set) F in X and each open covering $\mathscr{U}$ of X, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$, covering F, which is locally finite at each point of X (resp. at each point of $\bigcup\{V : V \in \mathscr{V}\}$.*

With these definitions, Grace proves the following two results:

(a) *A regular space is peripherally paracompact in the strong sense if and only if it is peripherally paracompact in the weak sense.*

(b) *A space is paracompact if and only if it is peripherally paracompact in the strong sense.*

6. D. R. Traylor [37] defines paracompactness in a discrete peripheral sense as below.

Definition. *A space X is said to be paracompact in a discrete peripheral sense if for every open covering $\mathscr{U}$ of X, there exists an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that* *if $\mathscr{F}$ be any discrete family of closed sets refining $\mathscr{V}$, then the boundary of $\bigcup\{F : F \in \mathscr{F}\}$ is paracompact with respect to the space X.*

The following result has been proved by Traylor:

Every regular semi-metric space which is paracompact in a discrete peripheral sense is paracompact.

7. H. Tamano [36] obtains a characterization of paracompactness in completely regular T_1 spaces. For this he makes use of the following definitions:

Definition. *A family of non-empty sets $\{U_\alpha : \alpha \in \Lambda\}$ with a well-ordered index set Λ is called a chain. A chain $\{U_\alpha : \alpha \in \Lambda\}$ is said to be a complete chain, with respect to a property $\mathscr{P}$, if $\bigcap\{U_\alpha : \alpha < \beta\}$ has the property $\mathscr{P}$ for each $\beta \in \Lambda$ whenever each U_α has the property $\mathscr{P}$.*

A descending chain of closed sets is always complete.

Tamano has proved the following result:

A completely regular T_1 space X is paracompact iff the following condition is satisfied: For every descending chain $\{F_\alpha : \alpha \in \Lambda\}$ of closed sets with empty intersection and for each complete chain $\{U_\alpha : \alpha \in \Lambda\}$ of open sets with $U_\alpha \supset F_\alpha$ (not necessarily having empty intersection) there is a descending complete chain $\{V_\alpha : \alpha \in \Lambda\}$ of open sets such that $F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$ and $\bigcap\{\overline{V}_\alpha : \alpha \in \Lambda\} = \emptyset$.

8. W. B. Sconyers has introduced the concept of hereditarily closure-preserving collections as follows:

Definition. *A family $\{F_\lambda : \lambda \in \Lambda\}$ of subsets of a space is said to be hereditarily closure-preserving if $\{F'_\lambda : \lambda \in \Lambda\}$ is closure-preserving, whenever $F'_\lambda \subset F_\lambda$ for each $\lambda \in \Lambda$.*

Sconyers has obtained the following characterization of paracompactness in regular spaces:

A regular space is paracompact iff every well-ordered open covering of X has a hereditarily closure-preserving closed refinement.

9. In another paper [35], Tamano introduces the concept of linearly cushioned refinements and again obtains a characterization of paracompactness for completely regular T_1 spaces.

Definition. *A family $\mathscr{V} = \{V_\lambda : \lambda \in \Lambda\}$ is said to be linearly cushioned in another family $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ if there exists a well-ordering of Λ and a mapping*

$\Phi : \Lambda \rightarrow \Delta$ such that $\overline{[\bigcup\{V_\lambda : \lambda \in \Lambda'\}]} \subset \bigcup\{U_\alpha : \alpha \in \Phi(\Lambda')\}$ for each bounded subset Λ' of Λ.

10. Tamano then proves the following result:

A completely regular T_1 space is paracompact iff every open covering has a linearly cushioned open refinement.

Tamano [34] has also introduced the notion of linearly locally finite families.

Definition. *Let $\mathscr{U}$ be a family of subsets of a space X and let $\leqq$ be a linear order on $\mathscr{U}$. A subfamily $\mathscr{U}'$ of $\mathscr{U}$ is said to be majorized if there exists a member U of $\mathscr{U}$ such that $U' \leqq U$ for every U' in $\mathscr{U}'$. A family $\mathscr{U}$ is said to be linearly locally finite with respect to $\leqq$ if every majorized subfamily of $\mathscr{U}$ is locally finite.*

Tamano has proved in his paper that a completely regular space is paracompact iff every open covering has a linearly locally finite open refinement. This result was improved by S. Swaminathan who, in an unpublished paper, introduced the concept of linearly closure-preserving collections as follows:

Definition. *A family $\mathscr{U}$ with a linear order $\leqq$ on it is said to be linearly closure-preserving if every majorized subcollection of it is closure-preserving.*

Swaminathan proved the following result:

For a regular space X, the following are equivalent:

(a) *X is paracompact.*

(b) *Every open covering of X has a linearly locally finite open refinement.*

(c) *Every open covering of X has a linearly closure-preserving open refinement.*

11. J. E. Vaughan [38] uses a different definition of linearly cushioned collections. If in Tamano's definition "well-ordering" is replaced by "linear ordering" and "bounded" by "majorized", then we get Vaughan's definition of a linearly cushioned collection. With this definition, Vaughan obtains the following result:

A regular space X is paracompact iff every open covering of X has a linearly cushioned open refinement.

12. As a generalization of the concept of linearly locally finite families, Y. Katuta [21] has introduced the notion of order locally finite families as below:

Definition. *A family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets of a space X is said to be order locally finite if there is a linear ordering $<$ of the index set Λ such that for each $\lambda \in \Lambda$, $\{U_u : u < \lambda\}$ is locally finite at each point of U_λ.*

Katuta has proved the following results:

(1) *A regular space is paracompact if and only if every open covering of X has an order locally finite open refinement.*

(2) *If a regular space X has two coverings $\{V_\lambda : \lambda \in \Lambda\}$ and $\{U_\lambda : \lambda \in \Lambda\}$ such that $\{U_\lambda : \lambda \in \Lambda\}$ is order locally finite, V_λ is compact, U_λ is open and $V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$, then, for any paracompact regular space Y, $X \times Y$ is paracompact.*

In another paper [22], Katuta has obtained some sufficient conditions for a space to be paracompact.

(1) *If $\{G_\lambda : \lambda \in \Lambda\}$ is an order locally finite open covering of a regular space X such that, for each λ, $\bar{G}_\lambda$ is paracompact, then X is paracompact.*

(2) *If there is an order locally finite open covering $\{G_\lambda : \lambda \in \Lambda\}$ of a regular (resp. collectionwise normal T_1) space X such that for each λ, the boundary of G_λ is compact (resp. paracompact) and G_λ is paracompact, then X is paracompact.*

(3) *If there is a closed covering $\{F_\lambda : \lambda \in \Lambda\}$ and an order locally finite open covering $\{G_\lambda : \lambda \in \Lambda\}$ of X such that for each $\lambda \in \Lambda$, $F_\lambda \subset G_\lambda$ and F_λ is paracompact, then X is paracompact.*

13. B. H. McCandless [25] has initiated a study of order paracompact spaces and has studied their relationship with paracompact spaces.

Definition. *A space X is said to be order paracompact if every open covering $\mathscr{U}$ of X has an open refinement $\mathscr{V}$ which is linearly ordered and is such that for each $V \in \mathscr{V}$, the family $\{V' : V' < V, V' \in \mathscr{V}\}$ is locally finite at each point of $\bar{V}$.*

McCandless proves the following results:

(1) *A regular T_1 space is order paracompact if and only if it is paracompact.*

(2) *Every regular order paracompact space is collectionwise normal.*

(3) *The product of an order paracompact space with a compact space is order paracompact.*

In view of a result of Katuta mentioned earlier, it follows easily that the result (1) of McCandless above, remains true without the assumption of the space being T_1. Also, then (2) is obvious, since every paracompact regular space is collectionwise normal.

We have obtained in [32] some results on order paracompact spaces. We have shown that order paracompactness is inversely preserved under mappings which are closed, continuous with point inverses compact. Result (3) of McCandless mentioned above follows as a corollary to this result.

14. E. Michael [26] proved that a space which is the union of a locally finite family of closed, regular and paracompact sets is paracompact. K. Morita [28] proved this result with "regular" replaced by "normal". We have proved in our paper [33] that this result remains true without the assumption of regularity or of normality. This is done by using a characterization of paracompactness obtained recently by John Mack and mentioned here earlier. Thus,

A space which is the union of a locally finite family of closed and paracompact sets is paracompact.

In the same paper, we obtain a general theorem and show that in view of the result obtained above, this general theorem is applicable to paracompactness and thus obtain the following result:

If $\mathcal{V}$ be an order locally finite open covering of a space X such that the closure of each member of $\mathcal{V}$ is paracompact, then X is paracompact.

As has been mentioned earlier, the above result was proved by Katuta with the additional assumption that the space is regular.

15. Motivated by the definition of order locally finite families of Katuta, we have introduced in [32] the concepts of order closure-preserving and order cushioned refinements.

Definitions. *Let $\mathcal{U}$ be a family of subsets of a space X well-ordered by $<$. $\mathcal{U}$ is said to be order closure-preserving if for every $U \in \mathcal{U}$ and every subfamily $\mathcal{U}'$ of $\{U' : U' < U\}$ we have*

$$\mathrm{Cl}_U[\bigcup\{U' \cap U : U' \in \mathcal{U}'\}] = \bigcup\{\mathrm{Cl}_U(U' \cap U) : U' \in \mathcal{U}'\}\,.$$

Here Cl_U denotes the closure in the relative topology of U. $\mathcal{U}$ is said to be order cushioned in another family $\mathcal{V}$ with a cushion map $f : \mathcal{U} \to \mathcal{V}$ if for every $U \in \mathcal{U}$ and every subfamily $\mathcal{U}'$ of $\{U' : U' < U\}$ we have,

$$\mathrm{Cl}_U[\bigcup\{U' \cap U : U' \in \mathcal{U}'\}] \subset \bigcup\{f(U') : U' \in \mathcal{U}'\}\,.$$

With these definitions, we have proved the following result:

For a regular space X, the following are equivalent:

(i) *X is paracompact.*

(ii) *Every open covering of X has an order locally finite (relative to a well-order) open refinement.*

(iii) *Every open covering of X has an order closure-preserving open refinement.*

(iv) *Every open covering of X has an order cushioned open refinement.*

16. As defined by K. Kuratowski [23], a set is called *regularly closed* if it is the closure of its own interior or equivalently, if it is the closure of an open set. Z. Frolík [14] calls a space *weakly regular* if every non-empty open subset contains a non-empty regularly closed set. With these definitions, we, in our paper [30], obtain the following characterizations of paracompactness:

(1) *A space X which contains a non-empty, proper regularly closed subset of X is paracompact iff every proper regularly closed subset of X is paracompact.*

In the "if" part of the above result, the condition that X contains a non-empty proper regularly closed subset cannot be dropped even under the stronger hypothesis that every proper closed subset of X is paracompact.

(2) *A weakly regular space X is paracompact iff every proper regularly closed subset of X is paracompact.*

(3) *A semi-regular space X is paracompact iff every proper regularly closed subset of X is paracompact.*

2. Countable Paracompactness

1. In [3], C. E. Aull proved that a T_2 space is metrizable iff it is countably paracompact and has a σ-locally finite base. He improves his result in [5] by replacing "countably paracompact" by "locally countably paracompact" which is a weaker form of countable paracompactness.

Definition. *A space X is said to be locally countably paracompact if each point of X has a countably paracompact neighbourhood. Here, a subset A of X is said to be countably paracompact if every countable open (in the original space X) covering of A has a locally finite (with respect to every point of X) open (in X) refinement.*

Definition. *A space is said to be an E_1 space if every point is the intersection of a countable number of closed neighbourhoods.*

Aull proves that every first axiom T_2 space is an E_1 space and that every locally countably paracompact E_1 space is T_3, thus proving that,

A T_2 space is metrizable iff it is locally countably paracompact and has a σ-locally finite base.

2. It is well-known that every normal space in which every closed set is a G_δ is countably paracompact (in fact, hereditarily countably paracompact). J. Greever [17] shows that "normal" here cannot be replaced by "completely regular T_1". As for the converse, Robert Briggs [9] remarks that if Ω be the first uncountable segment of the ordinals and Ω' be Ω together with its end point, then $\Omega \times \Omega'$ is a countably paracompact T_3 space which is not normal. J. N. Younglove in [6] raises the following problem:

Problem (1). *If X is a countably paracompact T_2 space in which every closed set is a G_δ, then is X normal?*

P. L. Zenor [40] proves that not every countably paracompact T_2 space is hereditarily countably paracompact and raises the following question:

Problem (2). *If X is a countably paracompact T_2 space in which every closed set is a G_δ, then is X hereditarily countably paracompact?*

Since every closed continuous image of a paracompact T_2 space is paracompact, it is natural to ask whether every closed continuous image of a countably paracompact T_2 space is countably paracompact. Again Zenor proves that not every closed continuous image of a countably paracompact T_2 space is countably paracompact and asks the following:

Problem (3). *Is every closed continuous image of a countably paracompact T_2 space with every closed set as a G_δ, countably paracompact?*

Zenor shows that all these three problems are equivalent. He introduces the notion of weak normality as follows:

Definition. *A space X is said to be weakly normal if for every decreasing sequence $\{H_i\}$ of closed sets with empty intersection and for every closed set H such that $H \cap H_i = \emptyset$, there is an integer n and an open set U containing H_n such that $\overline{U} \cap H = \emptyset$.*

The main result of Zenor can then be stated as below:

Let X be a countably paracompact T_2 space every closed subset of which is a G_δ. Then the following assertions are equivalent:

(1) *X is normal.*

(2) *X is weakly normal.*

(3) *X is hereditarily countably paracompact.*

(4) *If f is a closed continuous mapping of X onto Y, then Y is countably paracompact.*

Zenor also shows that all above propositions are equivalent if X is a hereditarily countably paracompact T_2 space with every closed set a G_δ.

3. C. E. Aull [4] calls a space D_1 if every closed set has a countable base for the open sets containing it. He proves that every regular D_1 space is normal. Also, since every first axiom Hausdorff space is E_1 and every locally countably paracompact, E_1 space is T_3, it follows that every locally countably paracompact, T_2, D_1 space is normal. Also, in every T_1, D_1 space, every closed set is a G_δ. Thus,

Every locally countably paracompact (and hence also every countably paracompact), T_2, D_1 space is normal every closed subset of which is a G_δ.

4. Concerning preservation of countable paracompactness under closed continuous mappings, Zenor obtains the following two results:

(1) *If f is a closed continuous mapping of a countably paracompact and weakly normal space X onto Y, then Y is countably paracompact and weakly normal.*

(2) *If f is a closed continuous mapping of a countably paracompact T_2 space X onto a first axiom space Y, then Y is countably paracompact.*

3. $\mathfrak{m}$-Paracompactness

1. K. Morita [27] and A. Giovanni [15] independently introduced the concept of $\mathfrak{m}$-paracompact spaces and obtained several properties of the same. Most of the characterizations which they obtained of $\mathfrak{m}$-paracompact spaces were for normal spaces. By making use of directed covers John Mack [24], improved some of these results and obtained characterizations of $\mathfrak{m}$-paracompactness without requiring the space to be normal. He proved that *each of the following properties is equivalent to $\mathfrak{m}$-paracompactness for any space X and any infinite cardinal* $\mathfrak{m}$:

(1) *X is countably paracompact and every open covering of X of cardinality $\leqq \mathfrak{m}$ has a σ-locally finite open refinement.*

(2) *Every directed open covering of X of cardinality $\leqq \mathfrak{m}$ has a locally finite open refinement.*

(3) *For every directed open covering $\mathcal{U}$ of X of cardinality $\leqq \mathfrak{m}$, there exists a locally finite open covering $\mathcal{V}$ such that $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

(4) *Every well-ordered open covering of X of cardinality $\leqq \mathfrak{m}$ has a locally finite open refinement.*

(5) *For every well-ordered open covering $\mathcal{U}$ of X of cardinality $\leqq \mathfrak{m}$, there exists a σ-locally finite open covering $\mathcal{V}$ such that $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

The following two results proved by Mack had been obtained earlier by Morita with the assumption that X is normal.

(1) *X is $\mathfrak{m}$-paracompact and countably compact iff it is $\mathfrak{m}$-compact.*

(2) *A completely regular space X is $\mathfrak{m}$-paracompact and pseudo-compact iff it is $\mathfrak{m}$-compact.*

Mack obtained the following results for subsets of $\mathfrak{m}$-paracompact spaces.

(1) *Every generalized co-zero subspace of an* $\mathfrak{m}$*-paracompact space is* $\mathfrak{m}$*-paracompact.*

(2) *Every normal (or countably paracompact) generalized* F_σ *subspace of an* $\mathfrak{m}$*-paracompact space is* $\mathfrak{m}$*-paracompact.*

2. W. B. Sconyers in his paper [29] obtains a characterization of $\mathfrak{m}$-paracompactness by making use of the following definition of linearly hereditarily closure-preserving families.

Definition. *A family* $\{F_\lambda : \lambda \in \Lambda\}$ *is said to be linearly hereditarily closure-preserving if* Λ *can be well-ordered in such a way that for each* $\lambda \in \Lambda$, *the family* $\{F_\gamma : \gamma < \lambda\}$ *is hereditarily closure-preserving.*

The following result is then proved:

A normal space X *is* $\mathfrak{m}$*-paracompact iff for every well-ordered open covering* $\{U_\alpha : \alpha \in \Lambda\}$ *of cardinality* $\leqq \mathfrak{m}$, *there is a linearly hereditarily closure-preserving open covering* $\{V_\alpha : \alpha \in \Lambda\}$ *such that* $\overline{V}_\alpha \subset U_\alpha$ *for each* $\alpha \in \Lambda$.

3. We now summarize some of our results obtained in [30, 31].

We have proved that *a space is normal and* $\mathfrak{m}$*-paracompact if and only if every open covering of cardinality* $\leqq \mathfrak{m}$ *has a refinement of any of the following types*:

(i) *cushioned*;
(ii) *open, cushioned*;
(iii) *open, σ-cushioned*;
(iv) *open, linearly cushioned*;
(v) *open, order cushioned.*

For the weakly regular spaces of Frolík, the following characterization has been obtained.

A weakly regular space X *is* $\mathfrak{m}$*-paracompact iff every proper regularly closed subset of* X *is* $\mathfrak{m}$*-paracompact.*

A similar result for normal spaces can be stated as follows:

A normal space X *which contains a proper non-empty regularly closed set is* $\mathfrak{m}$*-paracompact iff every proper regularly closed subset of* X *is* $\mathfrak{m}$*-paracompact.*

C. H. Dowker [11] proved that every perfectly normal paracompact space is hereditarily paracompact. This result was improved by R. E. Hodel [20] by replacing "perfectly normal" by "totally normal" a concept due to Dowker [12].

We have obtained a similar result for $\mathfrak{m}$-paracompact spaces. Thus,

Every totally normal (and hence also every perfectly normal) $\mathfrak{m}$*-paracompact space is hereditarily* $\mathfrak{m}$*-paracompact.*

Concerning separation properties in $\mathfrak{m}$-paracompact spaces, the following results have been proved.

(1) *If X is an* $\mathfrak{m}$*-paracompact T_2 space such that each point of X has a neighbourhood basis of cardinality* $\leqq \mathfrak{m}$, *then X is T_3.*

(2) *Every* $\mathfrak{m}$*-paracompact T_2 space in which every closed set has a base of cardinality* $\leqq \mathfrak{m}$ *for the open sets containing it is T_4.*

Defining a locally $\mathfrak{m}$-paracompact space as a space in which every point has an $\mathfrak{m}$-paracompact neigbourhood, we have proved that every normal T_1 locally $\mathfrak{m}$-paracompact space can be embedded in an $\mathfrak{m}$-paracompact space as an open subspace.

We have obtained two characterizations of $\mathfrak{m}$-paracompact spaces among k-spaces and sequential spaces.

(1) *A k-space X is normal and* $\mathfrak{m}$*-paracompact iff every open covering of X of cardinality* $\leqq \mathfrak{m}$ *has a compact-finite closed refinement.*

(2) *A sequential space X is normal and* $\mathfrak{m}$*-paracompact iff every open covering of X of cardinality* $\leqq \mathfrak{m}$ *has a cs-finite closed refinement.*

The following are two sufficient conditions for a space to be $\mathfrak{m}$-paracompact.

(1) *If a space X has the W-weak topology with respect to a* $\mathscr{K}$*-collection* $\mathscr{K}$ *and if every open covering of X of cardinality* $\leqq \mathfrak{m}$ *has a* $\mathscr{K}$*-finite closed refinement, then X is* $\mathfrak{m}$*-paracompact.*

(2) *If a normal space X has the W-weak topology with respect to a* $\mathscr{K}$*-collection and if every open covering of X of cardinality* $\leqq \mathfrak{m}$ *has a* $\mathscr{K}$*-finite open refinement, then X is* $\mathfrak{m}$*-paracompact.*

Analogous to the four weaker forms of paracompactness introduced by Boone, one may introduce four weaker forms of $\mathfrak{m}$-paracompactness ($\mathfrak{m}$-mesocompact etc.) by putting cardinality restrictions on the covers. Many of Boone's results can be easily extended to these more general classes of spaces. It may be noted that for $\mathfrak{m} = \aleph_0$, four weaker forms of countable paracompactness will become characterizations of countable paracompactness in normal spaces.

The author wishes to express her deep gratitude to Prof. M. K. Singal for his encouragement and advice, his valuable suggestions and many fruitful discussions during the preparation of the present survey.

References

[1] *A. Arhangel'skiĭ:* Bicompact sets and the topology of spaces. Soviet Math. *42* (1963), 561 to 564.

[2] *A. Arhangel'skiĭ:* Some types of factor mappings and the relations between classes of topological spaces. Soviet Math. *4* (1963), 1726—1729.

[3] *C. E. Aull:* A note on countably paracompact spaces and metrization. Proc. Amer. Math. Soc. *16* (1965), 1316—1317.

[4] *C. E. Aull:* Closed set countability axioms. Indag. Math. *28* (1966), 311—316.

[5] *C. E. Aull:* A certain class of topological spaces. Commentationes Mathematicae *11* (1967), 49—53.

[6] *R. H. Bing* and *R. J. Bean:* Editors: Topology Seminar Wisconsin, Annals of Mathematics Studies No. 64, Princeton, N. J. 1966.

[7] *J. R. Boone:* Some characterisations of paracompactness in k-spaces. Unpublished.

[8] *J. R. Boone:* A metrization theorem for developable spaces. Unpublished.

[9] *R. C. Briggs:* A comparison of covering properties in T_3 and T_4 spaces. Doctoral Dissertation.

[10] *R. Brown:* Ten topologies for $X \times Y$. Quart. J. Math. *14* (1963), 303—309.

[11] *C. H. Dowker:* An embedding theorem for paracompact metric spaces. Duke Math. J. *14* (1947), 639—645.

[12] *C. H. Dowker:* Inductive dimension of completely normal spaces. Quart. J. Math. *4* (1953), 267—281.

[13] *S. P. Franklin:* Spaces in which sequences suffice. Fund. Math. *57* (1965), 107—115.

[14] *Z. Frolík:* Generalizations of compact and Lindelöf spaces. Czech. Math. J. *9* (1959), 172—217.

[15] *A. Giovani:* Ricovrimenti aperti structure uniformi sopra uno spazio topologico. Annali di Mathematica pura ed. applicato. (1959), 319—390.

[16] *E. E. Grace:* Peripheral covering properties imply covering properties. Canad. J. Math. *20* (1968), 257—263.

[17] *J. Greever:* On spaces in which every closed set is a G_δ. Proc. Japan Acad. *43* (1967), 445—447.

[18] *J. Greever:* On some generalized compactness properties. Publications of the Research Institute for Mathematical Sciences Kyoto University *4* (1968), 39—49.

[19] *J. Greever:* On screenable topological spaces. Proc. Japan Acad. *44* (1968), 434—438.

[20] *R. E. Hodel:* Total normality and the hereditary property. Proc. Amer. Math. Soc. *17* (1966), 462—465.

[21] *Y. Katuta:* A theorem on paracompactness of product spaces. Proc. Japan Acad. *43* (1967), 615—618.

[22] *Y. Katuta:* On paracompactness and metrizability of spaces. Proc. Japan Acad. *43* (1967), 748—751.

[23] *K. Kuratowski:* Topologie I. Hafner, New York, 1958.

[24] *J. Mack:* Directed covers and paracompact spaces. Canad. J. Math. *19* (1967), 649—654.

[25] *B. H. McCandless:* On order paracompact spaces. Canad. J. Math. *21* (1969), 400—405.

[26] *E. Michael:* A note on paracompact spaces. Proc. Amer. Math. Soc. *4* (1953), 831—838.

[27] *K. Morita:* On spaces having the weak topology with respect to closed coverings II. Proc. Japan Acad. *30* (1954), 711—717.

[28] *K. Morita:* Paracompactness and product spaces. Fund. Math. *50* (1961), 223—236.

[29] *W. B. Sconyers:* A note on $\mathfrak{m}$-paracompact spaces. Unpublished.

[30] *M. K. Singal* and *S. P. Arya:* On $\mathfrak{m}$-paracompact spaces. Math. Ann. *181* (1969), 119—133.

[31] *M. K. Singal* and *S. P. Arya:* On $\mathfrak{m}$-paracompact spaces II. Unpublished.

[32] *M. K. Singal* and *S. P. Arya:* A note on order paracompactness. Bull. Austr. Math. Soc. *4* (1971), 273—279.

[33] *M. K. Singal* and *S. P. Arya:* Two sum theorems for topological spaces. Israel Jour. Math. *8* (1970), 155—158.
[34] *H. Tamano:* Note on paracompactness. J. Math. Kyoto University *1—3*, (1963), 137—143.
[35] *H. Tamano:* Linearly cushioned refinements. Unpublished.
[36] *H. Tamano:* A characterization of paracompactness. Unpublished.
[37] *D. R. Traylor:* Weak peripheral properties imply covering properties. J. London Math. Soc. *44* (1969), 509—514.
[38] *J. E. Vaughan:* Linearly ordered collections and paracompactness. Unpublished.
[39] *C. Wenjen:* Concerning paracompact spaces. Proc. Japan Acad. *43* (1967), 121—124.
[40] *P. L. Zenor:* On countable paracompactness and normality. Unpublished.

MAITREYI COLLEGE, NETAJI NAGAR, NEW DELHI, INDIA