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In this paper we characterize topologically the connected chains without end points. Various mathematicians have tried to find a set of necessary and sufficient conditions for the orderability of a topological space.[1]) But no complete solution seems to be available for a general case. Here we have studied one important particular case, namely when the order is "continuous". In this direction also not much has been done and I do not know of a complete characterization except the interesting work of Eilenberg [2][2]). But that work is quite different from ours. The main theorem of Eilenberg says: "*A topological connected space X can be ordered if and only if the subset of its square X^2 obtained by deleting the diagonal of points (x, x) is not connected*". Thus he goes outside the ordered space S while our characterization is in terms of the properties of S only. Lynn studies in [6] the orderability of subsets of real numbers while in [5] he considers only metric zero dimensional space. Thus, for the general case the situation is unsatisfactory even to-day.

The core result of this paper is the following Theorem 1 and its Corollary 1. Here the results alone are given without proofs; the complete paper will appear somewhere else. A few of the results we present here have been investigated also by other authors and we cite their papers also. However, our work is quite independent of them. I thank the referee for drawing my attention to some of these works.

Theorem 1. *A topological space X is a continuously ordered space without end points if and only if X is* (i) *connected* (ii) *locally connected,* (iii) *every point is a cut point, and* (iv) *there are no three pairwise disjoint segments* (*generated by cut points*).

Corollary 1. *A topological space X is homeomorphic to the real line iff it is separable and satisfies* (i), (ii), (iii) *and* (iv) *of Theorem* 1.

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[1]) All spaces are assumed to be Hausdorff.

[2]) Editor's note: See also H. Herrlich, Ordnungsfähigkeit zusammenhängender Räume, Fund. Math. *57* (1965), 305—311 and H. J. Kowalski, Kennzeichnung von Bogen, Fund. Math. *46* (1958), 103—107.

Definitions

Some of the concepts defined below are already known in the literature; but terminologies used are different and so, for the sake of clarity, we will mention definitions of terms frequently used in this paper.

1. *A linearly ordered set S is called continuously ordered* [4, p. 58] *if one of the following is true*:

(a) *Every section of S is a cut section,*

(b) *S is densely ordered and every set bounded above (below) has a supremum (infimum).*

It can be proved that the order of S is continuous iff the order topology is connected. So, a continuously ordered space is also sometimes called a connected chain.

2. *A point p of a connected space X is called a cut point* [9] *of X if $X \setminus p$ breaks up into two non-null, separated sets. i.e. $X \setminus p = A \cup B$ such that $A \neq \neq \emptyset \neq B$,*

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) = \emptyset$$

This division of $X \setminus p$ is called a partition of X (generated by p) and will be denoted by $X \setminus p = A/B$. A and B will be called segments of the partition.

3. *A partition $X \setminus p = A/B$ of X generated by p is called unique if $X \setminus p = M/N$ also $\Rightarrow M = A$ or B.*

4. *A cut point p of a connected space X is called a strong cut point if each of the segments A, B of the partition generated by p is connected.*

5. *A point x of a connected space X is said to separate two points p and q of X* [9] *if $X \setminus x$ is a union of two separated parts one containing p and the other containing q.*

6. *Two segments P_1, P_2 (generated by cut points) are called comparable if either $P_1 \subset P_2$ or $P_2 \subset P_1$.*

Hereafter in this paper S will denote a continuously ordered space with its order topology.

Scheme of Proof

Using the following Proposition 1 the necessary part easily follows.

Proposition 1. *S can not have three pairwise disjoint segments (generated by cut points).*

Proposition 2. *If x is a cut point of a connected topological space X, and $X \setminus x = P/N$ is a partition, then*

(i) $\bar{P} = P \cup x$, $\bar{N} = N \cup x$,

(ii) *x is a limit point of both P and N,*

(iii) *P, N are open in X,*

(iv) *$\bar{P}$ and $\bar{N}$ are connected,*

(v) *if y is any other cut point of X such that $y \in P$, and $X \setminus y = P_1/N_1$, then either $P_1 \subset P$ or $N_1 \subset P$.*

Proposition 3. *If a connected space has no three pairwise disjoint segments (generated by cut points) then every cut point of the space is a strong cut point.*

Proposition 4. *A cut point p of a connected space X is a strong cut point if and only if p generates a unique partition.*

Proposition 5. [1] *If A is a connected subset of S containing two points a, b of S, then it contains the whole interval (a, b).*

Now, by using Propositions 2, 3 and 4 we get that one of the two segments of the partitions generated by the points of X are comparable (we call them P-segment of a cut-point x) and hence are totally ordered under the inclusion relation. This will define a total ordering among points of X. Then using Propositions 2 and 5 it can be proved that the order topology and the original topology are homeomorphic, and the order induced in X is continuous.

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