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Some generalizations of paracompactness

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SOME GENERALIZATIONS OF PARACOMPACTNESS

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The concept of paracompactness was introduced by Dieudonné [8]. Commenting on the importance of this concept, the great Russian topologist P. S. Aleksandrov [1] writes: "The class of paracompact spaces is very likely the most important class of topological spaces defined in recent years". There is undoubtedly a lot of truth in what he has said, for the concept was soon taken up by many research-workers, and as time passes, more and more people are being attracted by it. During a quarter-century of its existence, a lot of work has been done on paracompactness and its generalizations. In the present survey, it is proposed to enumerate various generalizations of paracompactness and to present some of the work in this direction that is being done at Delhi.

A space is said to be paracompact if every open cover of the space admits of a locally finite open refinement. Generalizations of paracompactness may be obtained by imposing cardinality restrictions on the cover or by modifying the nature of the cover or by requiring a refinement to be of a different type, or by combinations of these. Of course, one could have generalizations in other ways too.

Generalizations of paracompactness by placing cardinality restrictions on the cover were obtained by C. H. Dowker, M. Katětov, A. Giovanni and K. Morita. In 1951, the British mathematician C. H. Dowker [9] and the Czech mathematician M. Katětov [32], independently introduced the class of countably paracompact spaces by requiring that every countable open cover of the space admits of a locally finite open refinement. Since then, a number of workers such as Aull [3], Hayashi [17], Horne [22], Iséki [24, 25, 26], Ishikawa [31], Kljušin [33], Mack [35, 37], Mansfield [38], Rudin [47], Swaminathan [62] and others have shown their interest in these spaces. A survey of the work done on countably paracompact spaces was presented by me at a talk delivered at the fifth annual conference of the Kanpur Mathematical Society in April, 1966. It is going to appear shortly in the Mathematics Student. The concept of $\mathfrak{m}$-paracompact spaces was also introduced by two mathematicians independently – this time an Italian, A. Giovanni [15] and a Japanese, K. Morita [40]. A space is said to be $\mathfrak{m}$-paracompact if every open cover of cardinality $\leqq \mathfrak{m}$ admits of a locally finite open refinement. Some work on these spaces has been done by J. Mack [36, 37], T. Ishii [29, 30], M. K. Singal and Shashi Prabha Arya [56, 57]. A survey of the work on $\mathfrak{m}$-paracompact spaces was presented by me at the 32nd annual conference of the Indian Mathematical Society held at Patiala in December, 1966. It will appear shortly in the Mathematics Student.

A generalization of paracompactness was introduced by K. Morita [41] in 1948 by requiring that every open cover of the space should have a star-finite open refinement. He called such spaces spaces with the star-finite property. In 1949, E. G. Begle [5] studied these spaces under the name "S-spaces". In 1950, S. T. Hu [23] called them hypocompact spaces. Since every hypocompact space is paracompact, the Russian Mathematician Yu. M. Smirnov, who has done valuable work on these spaces, called them strongly paracompact spaces. Some work on these spaces has also been done by Iséki [27, 28], Nagami [42, 43, 44], Ponomarev [46], Smirnov [60], Trnková [64, 65], Yasui [66–69] and others.

Another generalization of paracompact spaces was introduced by Arens and Dugundji [2] in 1950 by requiring that every open cover of the space should admit of a point-finite open refinement. They called the spaces characterized by this property metacompact. Since every paracompact space is metacompact, these spaces were relabelled by Russian mathematicians as pointwise paracompact spaces. Some authors call these spaces weakly paracompact. These spaces have been studied by Grace [16], Hayashi [18, 19], Heath [16, 20], Traylor [63] and others.

We shall now see as to how one could obtain other generalizations of paracompactness by considering some modifications in the defining properties of spaces described above.

1. Almost Paracompact Spaces

A space is said to be almost compact if every open covering admits of a finite subfamily, the closures of whose members cover the space. The definition of a paracompact space and that of an almost compact space suggest the following definition:

A space is said to be *almost paracompact* if for each open covering $\mathscr{U}$ of X, there exists a locally finite family $\mathscr{V}$ of open subsets of X which refines $\mathscr{U}$ (The phrase "$\mathscr{V}$ refines $\mathscr{U}$" will mean that each member of $\mathscr{V}$ is contained in some member of $\mathscr{U}$. It should be distinguished from the phrase "$\mathscr{V}$ is a refinement of $\mathscr{U}$" which means that $\mathscr{V}$ refines $\mathscr{U}$ and $\bigcup\{V: V\in\mathscr{V}\} = \bigcup\{U: U\in\mathscr{U}\}$) and the family of closures of whose members covers the space.

We have made a study of almost paracompact spaces in [56].

It is obvious that a space is almost paracompact if and only if for each open covering $\mathscr{U}$ of the space there exists a locally finite family $\mathscr{V}$ of open sets which refines $\mathscr{U}$ and the union of whose members is dense in the space.

Almost $\mathfrak{m}$-paracompact and almost countably paracompact (also named as lightly paracompact) spaces can be similarly defined.

An almost paracompact space may fail to be paracompact. For example, let X be an infinite set and let $p \in X$. Let $\mathscr{T}$ be the topology generated by the family $\{\{p, x\}: x \in X\}$. Then $(X, \mathscr{T})$ is almost paracompact but not paracompact. However, a regular space is paracompact if and only if it is almost paracompact.

In terms of regularly closed sets, almost $\mathfrak{m}$-paracompact spaces may be characterized as follows:

A space is almost $\mathfrak{m}$*-paracompact (almost paracompact) if and only if every proper regularly closed subspace of the space is almost* $\mathfrak{m}$*-paracompact (almost paracompact).*

In a short note [58], we have recently characterized almost countably paracompact spaces as follows:

A space is almost countably paracompact if and only if for every decreasing sequence $\{F_i\}$ *of closed sets such that* $F_i^0 \neq \emptyset$ *for all* i *and* $\bigcap F_i = \emptyset$, *there exists a decreasing sequence* $\{G_i\}$ *of open sets such that* $\bigcap \bar{G}_i = \emptyset$ *and* $F_i^0 \subset \bar{G}_i$ *for all* i.

In another short note [48], another characterization of almost countably paracompact spaces is obtained as below:

A space X *is almost countably paracompact if and only if for every decreasing sequence* $\{F_i\}$ *of closed subsets of* X *such that* $F_i^0 \neq \emptyset$ *for all* i *and* $\bigcap F_i \subset U$ *where* U *is an open set, there exists a decreasing sequence* $\{G_i\}$ *of open (closed) subsets of* X *such that* $F_i^0 \subset \bar{G}_i\ (G_i)$ *for all* i *and* $\bigcap \bar{G}_i\ (\bigcap G_i) \subset \bar{U}$.

In the same note, a sufficient condition for a space to be almost countably paracompact has also been obtained in the following form:

A space X *is almost countably paracompact if for every decreasing sequence* $\{A_i\}$ *of regularly open sets such that* $\bigcap A_i = \emptyset$ *there exists a decreasing sequence* $\{G_i\}$ *of open subsets of* X *such that* $\bigcap \bar{G}_i = \emptyset$ *and* $\bar{G}_i \supset A_i$ *for each* i.

Two sufficient conditions for a space to be almost $\mathfrak{m}$-paracompact are as follows:

(i) *Let* $\{U_\alpha: \alpha \in \Lambda\}$ *be a locally finite open covering of a space* X *such that each* $\bar{U}_\alpha$ *is almost* $\mathfrak{m}$*-paracompact (almost paracompact). Then* X *is almost* $\mathfrak{m}$*-paracompact (almost paracompact).*

(ii) *Let* $\{G_\alpha: \alpha \in \Lambda\}$ *be a covering of* X *by pairwise disjoint open sets. If each* G_α *is almost* $\mathfrak{m}$*-paracompact (almost paracompact), then* X *is almost* $\mathfrak{m}$*-paracompact (almost paracompact).*

As regards preservation properties, we have the following:

(i) *If* f *is a closed continuous open mapping of a space* X *onto a space* Y *such that* $f^{-1}(y)$ *is compact for each* $y \in Y$, *then* Y *is almost* $\mathfrak{m}$*-paracompact (almost paracompact) if* X *is so.*

(ii) *If* f *is a closed continuous irreducible mapping (that is, such that no proper closed subset of* X *maps onto* Y*) of a space* X *onto a space* Y *such that* $f^{-1}(y)$ *is*

$\mathfrak{m}$-compact (compact) for each $y \in Y$, then X is almost $\mathfrak{m}$-paracompact (almost paracompact) whenever Y is so.

With regard to products, almost paracompact spaces behave in the same way as paracompact spaces. The product of two copies of the reals each equipped with the lower limit topology is a regular non-paracompact space [61] and is therefore non-almost paracompact, whereas each of the factor spaces is almost paracompact.

However, we have the following:

If X is almost paracompact and Y is almost compact, then $X \times Y$ is almost paracompact.

As corollaries to the above result, we have the following:

(i) *The product of an almost paracompact space with a compact space is almost paracompact.*

(ii) *The product of a paracompact space with an almost compact space is almost paracompact.*

2. Nearly Paracompact Spaces

A new class of spaces which contains the class of compact spaces and is contained in the class of almost compact spaces has been introduced recently in [59]. A space is said to be nearly compact if every regular open covering of the space has a finite subcovering. This definition, taken together with that of paracompact spaces, suggests the following:

A space is said to be *nearly paracompact* if every regular open covering has a locally finite open refinement.

Nearly paracompact spaces have been introduced and studied in [53].

It can be easily seen that a space X is nearly paracompact if and only if for every open covering $\mathcal{U}$ of X there exists a locally finite family $\mathcal{V}$ of open subsets of X which refines $\mathcal{U}$ and the family $\{\overline{V}^0 : V \in \mathcal{V}\}$ covers X. In view of this characterization it is obvious that

$$\text{paracompact} \Rightarrow \text{nearly paracompact} \Rightarrow \text{almost paracompact}\,.$$

None of the reverse implications holds in general, as can be seen from the following examples:

(i) Let $X = \{(1/n, y)\colon n = 1, 2, 3, \ldots, 0 \leqq y \leqq 1\} \cup \{p\}$. The points $(1/n, y)$ possess usual neighbourhoods in the plane. A fundamental system of neighbourhoods of p consists of the sets G_m where $G_m = (U_m - A) \cup \{p\}$ where $U_m = \{(1/n, y)\colon (1/n, y) \in X \text{ and } n > m\}$ and A is countable. The space X is then a nearly paracompact Hausdorff space which is not paracompact.

(ii) Let $X = \{a_m, a_{mn}: m = 1, 2, \ldots, n = \pm 1, \pm 2, \ldots\} \cup \{p\} \cup \{q\}$ where $a_m = (1/m, 0)$, $a_{mn} = (1/m, 1/n)$. All points a_m and a_{mn} have usual neighbourhoods in the plane. The fundamental system of neighbourhoods of p consists of sets of the form $U^k(p) \cup \{p\}$ where $U^k(p) = \{a_{mn}: m > k, n > 0\}$ and that of q consists of sets of the form $U^k(q) \cup \{q\}$ where $U^k(q) = \{a_{mn}: m > k, n < 0\}$. Then X is a Hausdorff semi-regular almost paracompact space which is not nearly paracompact.

Also, every infinite discrete space is an example of a nearly paracompact space which is not nearly compact.

In a semi-regular space,

paracompact $\Leftrightarrow$ *nearly paracompact .*

Also, in an almost regular [52] or extremally disconnected space,

almost paracompact $\Leftrightarrow$ *nearly paracompact .*

In a lightly compact [4] space,

nearly compact $\Leftrightarrow$ *nearly paracompact .*

A number of characterizations of nearly paracompact spaces have been obtained for almost regular spaces.

An almost regular space is nearly paracompact if and only if every regular open covering has a refinement of any of the following types:

(i) *locally finite;*
(ii) *locally finite closed;*
(iii) *locally finite regularly closed;*
(iv) *locally finite regularly open.*

It can be shown by examples that the requirement of almost regularity cannot be replaced by Hausdorff condition or by semi-regularity.

Again, *an almost regular space is nearly paracompact if and only if each regular open covering has*

(i) *a σ-locally finite open refinement; or*
(ii) *a σ-discrete open refinement; or*
(iii) *a regularly open star-refinement; or*
(iv) *a locally finite partition of unity subordinated to it; or*
(v) *a partition of unity subordinated to it.*

For almost normal [54], T_1 spaces, the following has been obtained:

An almost normal T_1 space is nearly paracompact if and only if every regular open covering has a refinement of any of the following types:

(i) *open closure preserving*;

(ii) *open cushioned*;

Regarding separation properties in nearly paracompact spaces, the following results have been obtained:

(i) *Every nearly paracompact Hausdorff space is almost regular.*

(ii) *In a weakly regular* [52] *nearly paracompact space, every pair of disjoint regularly closed sets can be strongly separated.*

An open subspace of a nearly paracompact space may fail to be nearly paracompact. We believe that even a closed subspace of a nearly paracompact space might fail to be nearly paracompact. However, a subspace of a nearly paracompact space which is both open and closed is necessarily nearly paracompact.

Concerning mappings and nearly paracompact spaces, the following two results have been obtained:

(i) *If f is a closed almost continuous* [49] *and almost open* [49] *mapping of a nearly paracompact space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is nearly paracompact.*

(ii) *A space X is nearly paracompact if for each regular open covering $\mathscr{W}$ of X, there exists an $(\mathscr{W}, p)$-mapping of X into some paracompact space Y where p is the property of being locally finite.*

The product of two nearly paracompact spaces may fail to be nearly paracompact. However, we have the following:

The product of a nearly paracompact and a nearly compact space is nearly paracompact.

As corollaries to the above, we have the following two results:

(i) *The product of a nearly paracompact space with a compact space is nearly paracompact.*

(ii) *The product of a paracompact space with a nearly compact space is nearly paracompact.*

3. Mildly Paracompact Spaces

A space is said to be mildly compact if every countable regular open covering has a finite subcovering. We have introduced in [55] a new class of spaces which contains the class of mildly compact spaces and also that of countably paracompact spaces. This is defined as follows:

A space X is said to be *mildly paracompact* if every countable regular open covering of X has a locally finite open refinement.

It can be proved easily that a space is mildly paracompact if and only if for every countable open covering $\mathscr{U}$ of X, there exists a locally finite family of open sets which refines it and the family of the interiors of the closures of whose members covers X.

Obviously,

$$\text{Countably paracompact} \Rightarrow \text{mildly paracompact} \Rightarrow \text{lightly paracompact}$$

and

$$\text{mildly compact} \Rightarrow \text{mildly paracompact}\,.$$

The following examples show that the reverse implications do not hold in general.

(i) Let $X = \{a_{ij}, a_i, a\colon i, j = 1, 2, \ldots\}$. Let each point a_{ij} be isolated. Let $\{U^k(a_i)\colon k = 1, 2, \ldots\}$ be the fundamental system of neighbourhoods of a_i where $U^k(a_i) = \{a_i, a_{ij}\colon j \geqq k\}$ and let the fundamental system of neighbourhoods of a be $\{V^k(a)\colon k = 1, 2, \ldots\}$ where $V^k(a) = \{a, a_{ij}\colon i \geqq k, j \geqq k\}$. Then X is a Hausdorff non regular space which is mildly paracompact but not countably paracompact.

(ii) Example in [9] is an example of a normal, mildly paracompact space which is not countably paracompact.

(iii) Let $X = \{a_{ij}, b_{ij}, c_i, a\colon i, j = 1, 2, \ldots\}$. Let each a_{ij} and b_{ij} be isolated. Let the fundamental system of neighbourhoods of c_i be $\{U^n(c_i)\colon n = 1, 2, \ldots\}$ where $U^n(c_i) = \{c_i, a_{ij}, b_{ij}\colon j \geqq n\}$ and that of a be $\{U^n(a)\colon n = 1, 2, \ldots\}$ where $U^n(a) = \{a, a_{ij}\colon i, j \geqq n\}$. Then X is lightly paracompact but not mildly paracompact.

(iv) Consider the set X of all positive integers with the discrete topology. Then X is mildly paracompact but not mildly compact.

However, we have

$$\textit{mildly compact} \Leftrightarrow \textit{mildly paracompact} + \textit{lightly compact}\,.$$

Also, *in an extremally disconnected space,*

$$\textit{mildly paracompact} \Leftrightarrow \textit{lightly paracompact}\,.$$

For normal spaces, we have a large number of characterizations of mildly paracompact spaces. It has been proved that:

A normal space X is mildly paracompact if and only if every countable regular open covering $\{U_i\colon i = 1, 2, \ldots\}$ of X has a refinement of any of the following types:

(i) *point-finite regular open;*

(ii) *point-finite open;*

(iii) *open refinement $\{V_i\colon i = 1, 2, \ldots\}$ such that $\overline{V}_i \subset U_i$ for each i;*

(iv) *star-finite open;*

(v) *countable locally finite closed*;
(vi) *countable closure-preserving closed*;
(vii) *countable closed*;
(viii) *σ-discrete closed*;
(ix) *σ-locally finite closed*;
(x) *σ-closure preserving closed*;
(xi) *star-refinement open*;
(xii) *cushioned*;
(xiii) *σ-cushioned.*

It has been shown that *a normal space is mildly paracompact if and only if any of the following conditions hold*:

(i) *For every sequence $\{F_i\}$ of non-empty regularly closed sets such that $\bigcap F_i = \emptyset$, there is a sequence $\{G_i\}$ of open sets such that $\bigcap G_i = \emptyset$ and $F_i \subset G_i$ for each i.*

(ii) *For every sequence $\{F_i\}$ of non-empty regularly closed sets such that $\bigcap F_i = \emptyset$, there is a sequence $\{B_i\}$ of closed G_δ sets such that $\bigcap B_i = \emptyset$ and $F_i \subset B_i$ for each i.*

(iii) *Every countable regular open covering has a locally finite partition of unity subordinated to it.*

(iv) *Every countable regular open covering has a partition of unity subordinated to it.*

The following is a sufficient condition for a space to be mildly paracompact:

If for every sequence $\{F_i\}$ of non-empty regularly closed sets with empty intersection, there exists a decreasing sequence $\{G_i\}$ of open subsets of X such that $\bigcap \overline{G}_i = \emptyset$ and $G_i \supset F_i$ for each i, then X is mildly paracompact.

4. Strongly $\mathfrak{m}$-Paracompact Spaces

Strongly $\mathfrak{m}$-paracompact spaces have been introduced and studied in [51].

A space X is said to be *strongly $\mathfrak{m}$-paracompact* if every open covering of X of cardinality $\leq \mathfrak{m}$ has a star-finite open refinement.

It may be noted that a strongly $\aleph_\alpha$-paracompact space may fail to be strongly $\aleph_{\alpha+1}$-paracompact even if it be normal Hausdorff. For any ordinal α, the linearly ordered space $W(\omega_{\alpha+1})$ consisting of all ordinals less than the initial ordinal $\omega_{\alpha+1}$ of cardinality $\aleph_{\alpha+1}$ with the interval topology is an example of such a space.

A characterization of strongly $\mathfrak{m}$-paracompact spaces in normal spaces is given below:

A normal space X is strongly $\mathfrak{m}$-paracompact if and only if it is strongly countably paracompact and each open covering of X of cardinality $\leqq \mathfrak{m}$ has a star-finite open refinement.

An open subspace of a strongly $\mathfrak{m}$-paracompact space may fail to be strongly $\mathfrak{m}$-paracompact. Every closed subspace of a strongly $\mathfrak{m}$-paracompact space is however strongly $\mathfrak{m}$-paracompact.

Concerning subspaces, the following results have been obtained:

(i) *A subset A of a space X is strongly $\mathfrak{m}$-paracompact if and only if for each open set G containing A, there is a strongly $\mathfrak{m}$-paracompact subspace Y such that $A \subset Y \subset G$.*

(ii) *Every generalized F_σ (or generalized co-zero) subspace of a strongly paracompact space is strongly paracompact.*

(iii) *Let Y be a generalized F_σ subspace of a strongly $\mathfrak{m}$-paracompact space X. If Y is normal, then Y is strongly $\mathfrak{m}$-paracompact.*

(iv) *Every generalized F_σ (or generalized co-zero) subspace of a normal, strongly $\mathfrak{m}$-paracompact space is strongly $\mathfrak{m}$-paracompact.*

(v) *Every subspace of a perfectly normal, strongly $\mathfrak{m}$-paracompact space is strongly $\mathfrak{m}$-paracompact.*

(vi) *A space X is $\mathfrak{m}$-paracompact and locally strongly $\mathfrak{m}$-paracompact if it is a union of a locally finite family of open sets with strongly $\mathfrak{m}$-paracompact closures.*

(vii) *If a space X is paracompact and locally strongly $\mathfrak{m}$-paracompact, then it is the union of a locally finite family of open sets with strongly $\mathfrak{m}$-paracompact closures.*

Regarding inverse preservation of strongly $\mathfrak{m}$-paracompactness, the following result has been proved:

If f is a closed continuous mapping of a space X onto a strongly $\mathfrak{n}$-paracompact space Y such that $f^{-1}(y)$ is $\mathfrak{m}$-compact for each $y \in Y$, then X is strongly $\mathfrak{n}$-paracompact whenever $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals such that $\mathfrak{n} \leqq \mathfrak{m}$.

The product of two strongly paracompact spaces may fail to be strongly paracompact. However, the product of a strongly paracompact space with a compact space is strongly paracompact. A number of similar results have been proved for strongly $\mathfrak{m}$-paracompact spaces:

(i) *If X is a strongly $\mathfrak{n}$-paracompact space such that every point of X has a neighbourhood basis of cardinality $\leqq \mathfrak{m}$ and Y is $\mathfrak{m}$-compact, then $X \times Y$ is strongly $\mathfrak{n}$-paracompact whenever $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals with $\mathfrak{n} \leqq \mathfrak{m}$.*

(ii) *If X is a strongly countably paracompact, first axiom space and Y is countably compact, then $X \times Y$ is strongly countably paracompact.*

(iii) *If X is a strongly $\mathfrak{m}$-paracompact space and Y is compact, then $X \times Y$ is strongly $\mathfrak{m}$-paracompact.*

Using the notions of simple extension [34], invertibility [11] and generalized invertibility [21], following results have been obtained:

(i) *Let $(X, \mathscr{T})$ be any space and let $\mathscr{T}(A)$ be a simple extension of $\mathscr{T}$. If $X - A \in \mathscr{T}$, then $(X, \mathscr{T}(A))$ is strongly $\mathfrak{m}$-paracompact if and only if $(A, \mathscr{T} \cap A)$ and $(X - A, \mathscr{T} \cap (X - A))$ are strongly $\mathfrak{m}$-paracompact.*

(ii) *Let $(X, \mathscr{T})$ be a generalized invertible space and let (U, h) be an inverting pair for X. If $(X, \mathscr{T})$ is regular and $U \subset A$ where A is strongly paracompact, then X is strongly paracompact.*

(iii) *If $(X, \mathscr{T})$ is a regular invertible space and if U is an open subset of X which is strongly paracompact, then $(X, \mathscr{T})$ is strongly paracompact.*

(iv) *If $(X, \mathscr{T})$ is a generalized invertible space, (U, h) an inverting pair for X and A is a closed and regular subspace of X such that $U \subset A$, then X is strongly paracompact if and only if A is strongly paracompact.*

(v) *If $(X, \mathscr{T})$ be a normal strongly countably paracompact, generalized invertible space and (U, h) be an inverting pair for X, $U \subset A$ where A is strongly $\mathfrak{m}$-paracompact, then X is strongly $\mathfrak{m}$-paracompact.*

A space is said to be *almost strongly paracompact* if for each open covering $\mathscr{U}$ of X, there is a star-finite family $\mathscr{V}$ of open subsets of X which refines $\mathscr{U}$ and the family of the closures of whose members covers X.

Obviously, every strongly paracompact space is almost strongly paracompact. But an almost strongly paracompact space may fail to be strongly paracompact as can be seen from the following example:

Let $X = \{a, b, a_{ij}, b_{ij}, c_i \colon i, j = 1, 2, \ldots\}$. Let each point a_{ij} and b_{ij} be isolated. Let $\{U^k(c_i) \colon k = 1, 2, \ldots\}$ be the fundamental system of neighbourhoods of c_i where $U^k(c_i) = \{c_i, a_{ij}, b_{ij} \colon j \geqq k\}$ and let $\{V^k(a) \colon k = 1, 2, \ldots\}$ and $\{V^k(b) \colon k = 1, 2, \ldots\}$ be that of a and b respectively, where $V^k(a) = \{a, a_{ij} \colon i \geqq k,\ j = 1, 2, \ldots\}$ and $V^k(b) = \{b, b_{ij} \colon i \geqq k, j = 1, 2, \ldots\}$. Then X is almost strongly paracompact but not strongly paracompact.

Every strongly paracompact space is paracompact. It is not known, however, whether every almost strongly paracompact space is almost paracompact or not.

Following are some of the results proved about almost strongly paracompact spaces:

(i) *A space X is almost strongly paracompact if every proper regularly closed subset of X is almost strongly paracompact.*

(ii) *If each member of a pairwise disjoint open covering of a space is almost strongly paracompact, then the space is almost strongly paracompact.*

(iii) *The product of an almost strongly paracompact and an almost compact space is almost strongly paracompact.*

5. Pointwise $\mathfrak{m}$-Paracompact Spaces

Pointwise $\mathfrak{m}$-paracompact spaces offer a generalization of pointwise paracompact spaces.

A space X is said to be *pointwise* $\mathfrak{m}$-*paracompact* if every open covering of X of cardinality $\leqq \mathfrak{m}$ has a point-finite open refinement.

The following two characterizations of pointwise $\mathfrak{m}$-paracompact spaces have been obtained:

(i) *A space X is pointwise $\mathfrak{m}$-paracompact if and only if it is pointwise countably paracompact and each open covering of X of cardinality $\leqq \mathfrak{m}$ has a σ-point finite open refinement.*

(ii) *A T_1-space X is $\mathfrak{m}$-compact if and only if it is countably compact and pointwise $\mathfrak{m}$-paracompact.*

The following result which was proved by Morita with the assumption that X is normal, follows as a corollary to (ii) above.

A T_1-space X is $\mathfrak{m}$-compact if and only if it is countably compact and $\mathfrak{m}$-paracompact.

The following are two sufficient conditions for a space to be pointwise $\mathfrak{m}$-paracompact:

(i) *Let $\{G_\alpha \colon \alpha \in \Lambda\}$ be a family of subsets of a space X such that $\{G_\alpha^0 \colon \alpha \in \Lambda\}$ forms a point-finite open covering of X. If each G_α is pointwise $\mathfrak{m}$-paracompact, then X is pointwise $\mathfrak{m}$-paracompact.*

(ii) *Let X be a regular space and let $\mathscr{B}$ be an open basis of neighbourhoods of a point $x \in X$ such that $X - D$ is pointwise $\mathfrak{m}$-paracompact for each $D \in \mathscr{B}$, then X is pointwise $\mathfrak{m}$-paracompact.*

For a space to be pointwise paracompact, the following sufficient condition has been obtained:

A space X is pointwise paracompact if for every open covering $\mathscr{W}$ of X, there exists an $(\mathscr{W}, p)$-mapping of X onto some pointwise paracompact space Y where p is the property of being point-finite.

It has been proved that every closed continuous image of a pointwise countably paracompact space is pointwise countably paracompact.

For subspaces, we have the following two results:

(i) *Every generalized F_σ (or generalized co-zero) subspace of a pointwise $\mathfrak{m}$-paracompact space is pointwise $\mathfrak{m}$-paracompact.*

(ii) *Every subspace of a perfectly normal pointwise $\mathfrak{m}$-paracompact space is pointwise $\mathfrak{m}$-paracompact.*

6. $(\mathfrak{m}, \mathfrak{n})$-Paracompact Spaces

A space is said to be $(\mathfrak{m}, \mathfrak{n})$-compact [14] if every open covering of cardinality $\leqq \mathfrak{m}$ has a subcovering of cardinality $\leqq \mathfrak{n}$. Generalizing this concept as also that of paracompactness, the class of $(\mathfrak{m}, \mathfrak{n})$-paracompact spaces has been introduced and studied in [50].

A space X is said to be $(\mathfrak{m}, \mathfrak{n})$-*paracompact* if every open covering of X of cardinality $\leqq \mathfrak{m}$ has a locally-$\mathfrak{n}$ open refinement where a family $\mathscr{A}$ of subsets of X is said to be locally-$\mathfrak{n}$ if each point of X has a neighbourhood N such that the cardinality of the set $\{A \in \mathscr{A}: A \cap N \neq \emptyset\}$ is $\leqq \mathfrak{n}$.

Some types of $(\mathfrak{m}, \mathfrak{n})$-paracompactness are well-known. An $(\mathfrak{m}, 1)$-paracompact ("1" stands for finite cardinality) space is $\mathfrak{m}$-paracompact and an $(\aleph_0, 1)$-paracompact space is countably paracompact. An $(\infty, 1)$-paracompact ("∞" stands for infinite cardinality) space is nothing but a paracompact space.

The following characterizations of $(\mathfrak{m}, \mathfrak{n})$-paracompact spaces have been obtained:

(i) *If* $\{F_\alpha: \alpha \in \Lambda\}$ *be a locally finite family of closed subsets of* X *such that* $\bigcup\{F_\alpha^0: \alpha \in \Lambda\} = X$, *then* X *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact iff each* F_α *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(ii) *If* $\{F_\alpha: \alpha \in \Lambda\}$ *be a locally finite open covering of* X, *then* X *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact iff each* $\bar{F}_\alpha$ *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(iii) *A weakly regular space is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact iff every proper regularly closed subset of* X *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

Some sufficient conditions for a space to be $(\mathfrak{m}, \mathfrak{n})$-paracompact are obtained as below:

(i) *If a space* X *is countably paracompact and if every open covering of* X *of cardinality* $\leqq \mathfrak{m}$ *has a* σ-*locally*-$\mathfrak{n}$ *open refinement, then the space is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(ii) *If for every open covering* $\mathscr{W}$ *of* X *of cardinality* $\leqq \mathfrak{m}$ *there exists a finite* $\mathscr{W}$-*mapping of* X *onto some* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space* Y, *then* X *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(iii) *Let* X *be a regular space and let* x *be a point of* X *having a fundamental system of open neighbourhoods* $\mathscr{A}$ *with the property that* $X - A$ *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact for each* $A \in \mathscr{A}$. *Then* X *is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(iv) *If a countably paracompact normal space has a countable open covering by* $(\mathfrak{m}, \mathfrak{n})$-*paracompact sets, then the space is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

For subsets of $(\mathfrak{m}, \mathfrak{n})$-paracompact spaces, the following results have been obtained.

(i) *Every countably paracompact, generalized* F_σ *subspace of an* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(ii) *Every subspace of a totally normal* [10], *countably paracompact,* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(iii) *Every subspace of a perfectly normal* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

As regards inverse preservation of $(\mathfrak{m}, \mathfrak{n})$-paracompactness, the following result is proved:

If f is a closed continuous mapping of a space X onto an $(\mathfrak{m}, \mathfrak{n})$-*paracompact space Y, such that $f^{-1}(y)$ is* $(\mathfrak{m}, 1)$-*compact for each $y \in Y$, then X is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

From above follows a number of results regarding products.

(i) *If X is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact, Y is compact, then $X \times Y$ is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

From this result follows the result of Gál that the product of a compact space with an $(\infty, \mathfrak{n})$-compact space is $(\infty, \mathfrak{n})$-compact.

(ii) *If X is an* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space such that each point of X has a neighbourhood basis of cardinality* $\leqq \mathfrak{m}$ *and Y is an* $(\mathfrak{m}, 1)$-*compact space, then $X \times Y$ is* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

The classes of almost $(\mathfrak{m}, \mathfrak{n})$ and weakly $(\mathfrak{m}, \mathfrak{n})$-paracompact spaces have been introduced as generalizations of $(\mathfrak{m}, \mathfrak{n})$-paracompact spaces. Both these classes coincide with the class of almost $\mathfrak{m}$-paracompact spaces for $\mathfrak{n} = 1$.

A space X is said to be *almost* $(\mathfrak{m}, \mathfrak{n})$-*paracompact* if for each open covering $\mathscr{A}$ of X of cardinality $\leqq \mathfrak{m}$, there exists a locally-$\mathfrak{n}$ family $\mathscr{B}$ of open subsets of X which refines $\mathscr{A}$ and $\overline{[\bigcup\{B \colon B \in \mathscr{B}\}]} = X$. X is said to be *weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact* if for each open covering $\mathscr{A}$ of X of cardinality $\leqq \mathfrak{m}$, there exists a locally-$\mathfrak{n}$ family $\mathscr{B}$ of open subsets of X which refines $\mathscr{A}$ and $\bigcup\{\bar{B} \colon B \in \mathscr{B}\} = X$.

It is clear that every weakly $(\mathfrak{m}, \mathfrak{n})$-paracompact space is almost $(\mathfrak{m}, \mathfrak{n})$-paracompact. However, an example of an almost $(\mathfrak{m}, \mathfrak{n})$-paracompact space which is not weakly $(\mathfrak{m}, \mathfrak{n})$-paracompact is not known.

The following results have been obtained:

(i) *If every proper regularly closed subset of a space is weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact, then the space is weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(ii) *If $\{F_\alpha \colon \alpha \in \Lambda\}$ is a locally finite family of regularly closed weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact subsets of X which covers X, then X is weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(iii) *If f is a closed continuous and open mapping of a weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is weakly* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(iv) *If f is a closed continuous and open mapping of an almost* $(\mathfrak{m}, \mathfrak{n})$-*paracompact space X onto a space Y, such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is almost* $(\mathfrak{m}, \mathfrak{n})$-*paracompact.*

(v) *Every proper regularly closed subset of an almost* $(\mathfrak{m}, \mathfrak{n})$*-paracompact space is almost* $(\mathfrak{m}, \mathfrak{n})$*-paracompact.*

7. Generalizations Other than Our Own

J. N. Younglove [70] and R. C. Briggs [7] have introduced three generalizations of paracompactness.

Younglove defines spaces with the property Q in the following manner:

A space X is said to have the *property* Q if for each open covering $\mathscr{G}$ of X, there exists an open refinement $\mathscr{H}$ such that if $\{H_i: i = 1, 2, \ldots\}$ be a countably infinite subcollection of distinct elements of $\mathscr{H}$ and if $p_i, q_i \in H_i$ for each i and $\{p_i\} \to p$, then $\{q_i\} \to p$.

Briggs defines strong cover compact and weak cover compact spaces as below:

A space X is said to be *strong cover compact* if for each open covering $\mathscr{G}$ of X, there exists an open refinement $\mathscr{H}$ of $\mathscr{G}$ such that if $\{H_i: i = 1, 2, \ldots\}$ be any countably infinite subcollection of distinct elements of $\mathscr{H}$ and if $p_i, q_i \in H_i$ for each i, $p_i \neq p_j$, $q_i \neq q_j$ for $i \neq j$ and the point set $\{p_i: i = 1, 2, \ldots\}$ has a limit point in X, then the set $\{q_i: i = 1, 2, \ldots\}$ also has a limit point in X.

A space X is said to be *weak cover compact* if for each open covering $\mathscr{G}$ of X, there exists an open refinement $\mathscr{H}$ of $\mathscr{G}$ such that if $\{H_\alpha: \alpha \in \Lambda\}$ be any uncountable subcollection of distinct elements of $\mathscr{H}$ and if p_α and $q_\alpha \in H_\alpha$ for each α, $p_\alpha \neq p_\beta$, $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$ and the point set $\{p_\alpha: \alpha \in \Lambda\}$ has a limit point in X, then so does $\{q_\alpha: \alpha \in \Lambda\}$.

Each of the above properties is a generalization of paracompactness in as much as each is implied by paracompactness.

With regard to subspaces and products, strong cover compact, weak cover compact and spaces with property Q behave in the same way as paracompact spaces, that is, every closed subspace of a space possessing any one of these properties also possesses the same and the product of two spaces having these properties may fail to have the same.

Relationships between these three classes of spaces will become clear from the following:

(i) *In a first countable space,*

$$\textit{property } \mathrm{Q} \Rightarrow \textit{paracompact.}$$

(ii) *In a first countable* T_1 *(or a locally compact* T_1*) space,*

$$\textit{strong cover compact} \Rightarrow \textit{weak cover compact.}$$

(iii) *In a semi-metric T_3 or a developable T_3 space*

strong cover compact $\Leftrightarrow$ *weak cover compact* $\Leftrightarrow$ *property* Q $\Leftrightarrow$ *paracompact.*

(iv) *In a first countable T_3 or a locally compact T_3 space,*

property Q + *strong cover compact* $\Leftrightarrow$ *property* Q + *weak cover compact* $\Leftrightarrow$
$\Leftrightarrow$ *pointwise paracompact* + *strong cover compact* $\Leftrightarrow$
$\Leftrightarrow$ *pointwise paracompact* + *weak cover compact* $\Leftrightarrow$ *paracompact.*

(v) *In a first countable Lindelöf T_2 space,*

strong cover compact $\Leftrightarrow$ *paracompact.*

(vi) *In a T_3 space,*

property Q + *collectionwise normality* $\Leftrightarrow$ *pointwise paracompact* +
+ *collectionwise normality* $\Leftrightarrow$ *paracompactness.*

Briggs has constructed several examples to indicate the extent to which the conditions in the above implications can be weakened. Also, he has mentioned several unsolved problems some of which are given below:

(i) *Is every first countable, strong cover compact T_3 space a T_4 space?*

(ii) *Is every first countable, strong cover compact, separable T_3 space a T_4 space?*

(iii) *Is every first countable, separable, strong cover compact T_4 space paracompact?*

(iv) *Is the product of a compact space X and a first countable strong cover compact space strong cover compact?*

(v) *Is the image of a strong (weak) cover compact space under a closed mapping strong (weak) cover compact?*

R. M. Ford [13] introduces the concept of totally paracompact spaces.

A space is said to be *totally paracompact* if every open base has a locally finite subfamily covering the space.

Briggs calls such spaces basically paracompact.

Obviously, totally paracompact $\Rightarrow$ paracompact. But there exist paracompact spaces which are not totally paracompact.

Recently, K. Nagami has introduced the notion of σ-totally paracompact spaces.

A space X is said to be *σ-totally paracompact* if for every open base $\mathscr{B}$ of X, there exists a σ-locally finite open covering $\mathscr{A}$ of X such that for each $A \in \mathscr{A}$, there exists $B \in \mathscr{B}$ such that $A \subset B$ and boundary $A \subset$ boundary B.

Every regular σ-totally paracompact space is paracompact and every subspace of a totally normal σ-totally paracompact space is paracompact.

If in the above definition of σ-totally paracompact spaces, "σ-locally finite" be replaced by "order locally finite" ($\mathscr{A}$ is said to be order locally finite if there is a linear ordering "$<$" in $\mathscr{A}$ such that for each $A \in \mathscr{A}$, $\{A' \colon A' < A\}$ is locally finite at each point of A), then we get the definition of *order totally paracompact* spaces of B. Fitzpatrick, Jr. and R. M. Ford, [19].

Obviously, totally paracompact $\Rightarrow$ σ-totally paracompact $\Rightarrow$ order totally paracompact.

Ford [13] has proved that large and small inductive dimensions coincide in totally paracompact metric spaces. Nagami proved it to be true for totally normal, σ-totally paracompact spaces and Fitzpatrick and Ford have shown that this holds in every order totally paracompact metric space.

As a generalization of order totally paracompact spaces, B. H. McCandless has introduced the class of order paracompact spaces. For a brief discussion of order paracompact spaces, the reader is referred to a survey article by Shashi Prabha Arya which appears in these proceedings.

Some generalizations of paracompact spaces have been introduced by J. R. Boone [6] all of which become characterizations of paracompactness in certain classes of k-spaces.

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