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In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971. pp. 49--53.

Persistent URL: http://dml.cz/dmlcz/700564

PARACOMPACT AND COUNTABLY PARACOMPACT SUBSETS

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This paper is a summary of results of the author concerning various types of paracompact and countably paracompact subsets in references [1] through [4]. A few new results are also included, especially in regard to closure preserving and cushioned refinements.

Definition 1. *A subset M of a topological space $(X, \mathscr{T})$ is α-paracompact (σ-paracompact) if every open cover by members of $\mathscr{T}$ has an open locally finite (σ-locally finite) refinement by members of $\mathscr{T}$.*

Definition 2. *A subset M of a topological space $(X, \mathscr{T})$ is α-countably paracompact if every countable open cover by members of $\mathscr{T}$ has an open locally finite refinement by members of $\mathscr{T}$.*

In the above definitions the refinements are locally finite or σ-locally finite with respect to all points of X and not just points of M.

Definition 3. *A subset M of a topological space is β-paracompact (β-countably paracompact) if M is a paracompact (countably paracompact) subspace.*

The following definition is also useful.

Definition 4. *A subset M of a topological space is α-collectionwise normal if for every discrete family $\{D_a\}$, $D_a \subset M$, there is a pairwise disjoint family of open sets $\{G_a\}$ such that $D_a \subset G_a$ for every a.*

Again in the last definition the terms discrete and open refer to the topology of the space and not to the relative topology of M. Clearly every subset of a collectionwise normal space is α-collectionwise normal. For relations between α- and β-collectionwise normal subsets (collectionwise normal subspaces) see [2].

Summary and Some Related Results

In [3] it was shown that the following relations are satisfied for a subset M of a topological space $(X, \mathscr{T})$

(1) In a regular space α-P $\to$ σ-P $\to$ β-P.
(2) In any topological space α-CP $\to$ β-CP.
(3) In a normal space with M closed β-CP $\to$ α-CP.
(4) In a topological space σ-P + α-CP $\leftrightarrow$ α-P.
(5) In a regular normal space for M closed σ-P $\to$ α-P.
(6) In a regular normal space β-P + α-CN + generalized $F_\sigma \leftrightarrow$ σ-paracompactness (note the proof from left to right is given in [2]).

In the above CN, CP, and P stand for collectionwise normal, countably paracompact, and paracompact respectively.

In (3) and (5) we may replace "closed" by "generalized closed".

Definition 5. *A subset M of a topological space is a generalized closed set if for every open set G such that $M \subset G$ there exists a closed set F such that $M \subset \subset F \subset G$.*

Clearly in a T_1 space every generalized closed set is closed.

Theorem 1. *A generalized closed subset of a compact,* CP, P *space is compact,* α-CP, α-P *respectively.*

Theorem 2. *Let M be a dense generalized closed β-*CP *subset of a normal space $(X, \mathscr{T})$. Then $(X, \mathscr{T})$ is* CP.

Proof. We first prove that M is a normal subspace. The intersection of a closed and a generalized closed subset is a generalized closed subset. So if H is closed in M, then H is a generalized closed subset of X. Let A and B be closed in M and disjoint. Then $A = M \cap F$ where F is closed in X. Since B is a generalized closed set there exists a $\mathscr{T}$-closed set C, $B \subset C \subset \sim F$. The normality of M follows. Let $\{U_n\}$ be a countable open cover of X. Let $\{H_n\}$ be a countable relatively closed locally finite refinement of $\{M \cap U_n\}$. $\{H_n\}$ is locally finite with respect to all points of X as well as those of M. For if $a \in \sim M$, then there exists $b \in M$ such that $b \in \bar{a}$, since X is the only closed and hence the only open set containing M. There exists a neighborhood of b and hence a neighborhood of a intersecting only a finite number of members of $\{H_n\}$. For each n, $\sim \overline{H}_n \cup U_n = X$ since M is dense and a generalized closed set. So $\{\overline{H}_n\}$ is a $\mathscr{T}$-closed refinement of $\{U_n\}$.

Corollary 2. *For M a generalized closed set of a normal (normal regular) space β-*CP $\to$ α-CP, (σ-P $\to$ α-P).

Theorem 3. *In a regular paracompact (compact) space every* α-P *(compact) subset is a generalized closed set.*

Proof. Similar to Theorem 11 of [3].

Corollary 3A. *In a pseudometrizable or regular paracompact (compact) space the generalized closed sets and the* α-P *(compact) subsets are identical.*

Corollary 3B. *In a* T_2 *paracompact space the* α-P *and closed sets are identical.*

More generally it was shown in [3] that a closed subset of the interior of a closed β-P subset is α-P.

Recently Alo and Shapiro have proved a related result.

Theorem 4. *If* F *is an* α-P *subset of the closed subspace* S *of* $(X, \mathscr{T})$ *and if there exists an open set* G *in* X *such that* $F \subset G \subset S$ *then* F *is* α*-paracompact in* X.

Closure Preserving and Cushioned Refinements

Michael introduced closure preserving refinements in [6] and cushioned refinements in [7] in connection with equivalent formulations of paracompactness.

Theorem 5. *For regular normal space* $(X, \mathscr{T})$ *the following are satisfied for a subset* M.

$$(a) \leftrightarrow (b) \leftrightarrow (c) \leftarrow (d) \leftrightarrow (e) \leftrightarrow (f)$$

If M *is a generalized closed set* (c) → (d).

(a) *M is σ-paracompact.*

(b) *every* $\mathscr{T}$*-open cover of* M *has an* $(X, \mathscr{T})$ *σ-closure preserving* $\mathscr{T}$*-open refinement.*

(c) *every* $\mathscr{T}$*-open cover of* M *has an* $(X, \mathscr{T})$ $\mathscr{T}$*-open σ-cushioned refinement. The closures in the definition of cushioned refinement are* $\mathscr{T}$*-closures.*

(d) *M is α-paracompact.*

(e) *every* $\mathscr{T}$*-open cover of* M *has a* $\mathscr{T}$*-open* $(X, \mathscr{T})$ *closure preserving refinement.*

(f) *every* $\mathscr{T}$*-open cover of* M *has a* $\mathscr{T}$*-open cushioned refinement. Again as in (c) closures are* $\mathscr{T}$*-closures.*

Proof. (a) → (b) → (c), (d) → (e) → (f) → (c), are immediate. Similar to Theorems 8 and 11 of [3], we can show that M is α-CN and a generalized F_σ if it satisfies (c). By a result of Michael [7] M is β-P and is hence σ-P by (6) above, so

(c) → (a). Similar to Theorem 3 we can show that a set satisfying (f) is a generalized closed set in which case (a) → (d) by Corollary 2.

Some Properties of α-Countably Paracompact Subsets

An E_1 space is a topological space such that every point is the intersection of a countable number of closed neighborhoods.

Theorem 6. *An α-countably paracompact subset of an E_1 space is closed.*

Proof. See [1].

Corollary 6A. *Every countably compact E_1 space is maximally countably compact and minimally E_1.*

Proof. See [1].

Definition 6. *A topological space is locally countably paracompact if every point has an α-countably paracompact neighborhood.*

Corollary 6B. *Every locally countably paracompact E_1 space is T_3.*

Proof. See [1].

Corollary 6C. *A T_2 space is metrizable if and only if it is locally countably paracompact and has a σ-locally finite base.*

Theorem 7. *A generalized F_σ, α-countably paracompact subset of a T_4 space $(X, \mathscr{T})$ is closed.*

Proof See [4].

Clearly α-countably paracompact subset may be replaced by countably compact subset in the above theorem.

Theorem 8. *Let Z be a zero subset and M an α-countably paracompact subset of a topological space $(X, \mathscr{T})$ such that $Z \cap M = \emptyset$. Then Z and M are strongly separated.*

Proof. Z is the countable intersection of open sets V_n such that $Z = \bigcap \overline{V}_n$. $\{\sim \overline{V}_n\}$ is a countable open cover of M which has a locally finite open refinement $\{W_n\}$ such that $W_n \cap Z = \emptyset$. Let $W = \bigcup W_n$. Then $\sim \overline{W}$ and W are disjoint open sets containing Z and M respectively.

Corollary 8. *Let Z be a zero subset and F a closed subset of a countable paracompact space such that $Z \cap F = \emptyset$. Then Z and F are strongly separated.*

References

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