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MAPS OF EXTREMALLY DISCONNECTED SPACES, THEORY OF TYPES, AND APPLICATIONS

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Recall that an extremally disconnected space, shortly an ED-space, is a regular space such that the closure of each open set is open. Evidently every ED-space is zero dimmensional, and hence uniformizable. For brevity, if not otherwise stated, by a space we shall mean a separated uniformizable space (the assumption that the spaces are separated is irrelevant, but convenient).

There is a duality between Boolean algebras and compact totally disconnected spaces, and furthermore, a Boolean algebra B is complete if and only if the structure space of B (i.e. the Stone space of B) is an ED-space, and B is σ-complete if and only if the structure space of B is basically disconnected.

In section 1 we recall basic facts about ED-spaces, that might be useful in reading of this survey.

Our results on fixed points of mappings of ED-spaces in Sections 2 and 3 can be stated in terms of complete Boolean algebras, and in addition, the proofs translate from our language to the other one. We shall work with ED-spaces; the reader is invited to restate the results and proofs in terms of various concepts of Boolean Algebra Theory. The main result is that the set of all fixed points of a homeomorphism of a compact ED-space into itself is open.

In Section 4 the results of Section 2 and 3 are applied to the theory of types of free ultrafilters on countable sets.

The last two sections deal with applications to proofs of non homogeneity of closed sets in ED-spaces, and of $\beta P - P$ for spaces P that are not pseudocompact.

1. Elementary Properties of ED-spaces

In 1.1 – 1.4 we recall some well-known results on ED-spaces; if the reader finds it difficult to prove them, we invite him to consult the first chapters of [12].

1.1. *Any Čech-Stone compactification of an ED-space is an* ED-*space, a dense subspace of an* ED-*space is an* ED-*space, and every compact* ED-*space is a Čech-Stone compactification of each of its dense subspaces.*

1.2. *Every discrete space is an* ED-*space, and therefore Čech-Stone compactification of a discrete space is an* ED-*space. The Čech-Stone compactifications of discrete spaces are called free* ED-*spaces (they are free objects of the category of compact* ED-*spaces). We may think about free compact* ED-*spaces as ultrafilter spaces, i.e. the structure spaces of Boolean algebras of all subsets of sets, see Section* 4.

1.3. *Open subspaces of* ED-*spaces are* ED-*spaces, however a closed subspace of an* ED-*space need not be any* ED-*space, e.g.,* $\beta\mathsf{N} - \mathsf{N}$ *is no* ED-*space where* N *is the discrete space of natural numbers. The product of two* ED-*spaces need not be any* ED-*spaces, e.g.* $\beta\mathsf{N} \times \beta\mathsf{N}$ *is not any* ED-*space.*

1.4. *The statement that the closure of each open set is open is equivalent to the statement that if* V *and* U *are disjoint open sets, then* $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$.

1.5. *A space is said to be basically disconnected if the closure of each co-zero set is open. The dual concept for Boolean algebras is* σ*-complete.*

Remark. The results in 1.6 – 1.9 hold for spaces that admit an embedding as a closed subspace into a basically disconnected space. We shall make use of this remark in Sections 4 and 5.

Two sets X and Y in a space are semi-separated if $(X \cap \operatorname{cl} Y) \cup (\operatorname{cl} X \cap Y) = \emptyset$.

1.6. *Any two countable semi-separated sets* X *and* Y *in an* ED-*space* P *are functionally separated.*

Proof. There exist open sets $U \supset X$ and $V \supset Y$ with $U \cap V = \emptyset$. The characteristic function of $\operatorname{cl} U$ is continuous and separates X and Y.

As an immediate consequence we get:

1.7. *Any discrete countable set in an* ED-*space is normally embedded (that means that any bounded continuous function on the set admits an extension on the whole space).*

In consequence, no non-trivial sequence in an ED-space is convergent, and hence every metrizable ED-space is discrete. (Of course, every discrete space is a metrizable ED-space.) If X is a set in a space, denote by X^* the set $\operatorname{cl} X - X$.

1.8. Proposition. ([9], Lemma 1). *Assume that* X *and* Y *are discrete countable sets in an* ED-*space* P. *The set*

$$Z = (X \cap Y) \cup (X^* \cap Y) \cup (X \cap Y^*)$$

is discrete and normally embedded in P, and

$$\mathrm{cl}\, Z = \mathrm{cl}\, X \cap \mathrm{cl}\, Y, \quad Z^* = X^* \cap Y^* .$$

Proof. Evidently the inclusions $\subset$ hold. Clearly $\mathrm{cl}\, X \cap \mathrm{cl}\, Y \subset \mathrm{cl}\, Z \cup Z_1$, $X^* \cap Y^* \subset Z^* \cup Z_1$, where

$$Z_1 = \mathrm{cl}\,(X - \mathrm{cl}\, Y) \cap \mathrm{cl}\,(Y - \mathrm{cl}\, X) .$$

The set

$$Z_0 = (X - \mathrm{cl}\, Y) \cup (Y - \mathrm{cl}\, X)$$

is discrete, hence normally embedded, and therefore $Z_1 = \emptyset$.

As an immediate consequence we get the following result that will be needed in Section 4:

1.9. Proposition. *Let X and Y be two disjoint countable sets in an* ED-*space, and let $x \in \mathrm{cl}\, X \cap \mathrm{cl}\, Y$. Then either $x \in \mathrm{cl}\,(X^* \cap Y)$ or $x \in \mathrm{cl}\,(X \cap Y^*)$.*

2. Decomposable Sets

2.1. *Let f be a mapping of a space P into itself. A set $X \subset P$ is called f-invariant or invariant* wrt *f if $f[X] \subset X$. A set $Y \subset P$ is f-coinvariant if $P - Y$ is f-invariant (or equivalently, if $f^{-1}[Y] \subset Y$). A set is bi-invariant if it is simultaneously invariant and coinvariant.*

2.1. Definition. *Let f be a mapping of a space P into itself, and let $k \geqq 2$ be an integer. Call a set $B \subset P$, k-decomposable* wrt *f if B is the union of a disjoint family $\{B_i \mid i = 1, \ldots, k\}$ of closed-open sets B_i such that $f[B] \subset B$, and $B_i \cap \cap f[B_i] = \emptyset$ for all i.*

2.2. Lemma. *Let f be a continuous mapping of an* ED-*space E into itself, and let $k \geqq 2$ be an integer. There exists a largest k-decomposable set* wrt *f. the largest k-decomposable set is coinvariant, hence bi-invariant.*

This lemma is proved by considering the collection m of all k-decomposable sets, and showing, step by step, that:

a) The collection m is closed under disjoint unions (Boolean), and hence there exists a Z in m such that if $M \in m$, and if $M \cap Z = \emptyset$ then $M = \emptyset$.

b) Each element Z of m is contained in the smallest coinvariant set $Z_0 \supset Z$, and Z_0 belongs to m.

c) Take Z as in a; then Z_0 in b is the largest k-decomposable set.

2.3. Let f be a mapping of a set P into itself. If we endow P with the discrete topology then f becomes a continuous mapping of an ED-space into itself, and Lemma 2.2 applies. It is easy to verify that the complement Y of the largest 3-decomposable set is the smallest coinvariant set containing the set F of all fixed points, in particular, P is 3-decomposable if $F = \emptyset$. This is the essential part of Lemma on three sets, see Proposition 1 in [14], for a historical comment see [15], Remark 1.14.

The most convenient form of the Lemma for applications in the theory of types reads as follows:

Lemma on three sets. *Let f be a mapping of a set P into itself and let F be the set of all fixed points of f. Then $P - F$ is a disjoint union of three sets M_1, M_2 and M_3 such that $f[M_i] \cap M_i = \emptyset$.*

There is a generalization of 2.2 in the direction of Lemma on three sets. It should be remarked that this proposition will not be needed.

Proposition. *Let f be a continuous mapping of an* ED-*space E into itself, and let U be an open subset of E. There exists the largest U-relative k-decomposable set* wrt f.

By a U-relative k-decomposable set we mean a closed-open set B in U such that B is a disjoint union of a family $\{B_i \mid i = 1, \ldots, k\}$ of closed-open sets in U such that $f[B_i] \subset (E - U) \cup (B - B_i)$ for each i. The proof is the same.

In the case of a mapping of a discrete space P into itself we deduce from the Proposition that $P - F$, F being the set of all fixed points, is $(P - F)$-relative k-decomposable, which is precisely Lemma on three sets.

2.4. Existence Lemma. *If K is a compact* ED-*space and f is a homeomorphism of K into itself such that $fx \neq x$ for some x in K, then there exists a non-void 3-decomposable set.*

Proof. Put $K_1 = f[K]$, and choose a non-void closed-open set C_3 such that $f[C_3] \cap C_3 = \emptyset$. By induction choose closed open sets Z_n. $n = 1, 2, \ldots$, such that $Z_1 \cap C_3 = \emptyset$, $Z_1 \cap K_1 = f[C_3]$, and

$$Z_{n+1} \cap (C_3 \cup Z_1 \cup \ldots \cup Z_n) = \emptyset\,,$$
$$Z_{n+1}K_1 = f[Z_n] - C_3\,.$$

Let C_1 be the closure of the union of all Z_n with n odd, and C_2 the closure of the union of all Z_n with n even. It is easy to see that $\{C_i \mid i = 1, 2, 3\}$ is a 3-decomposition of $U\{C_i\}$.

Remark. Existence Lemma holds for basically disconnected compact spaces. On the other hand, 1.2 does not hold for basically disconnected compact space, see [10], Example.

3. Fixed Points

Applying Lemma 1.4 to the complement of the largest 3-decomposable set (Lemma 1.2) we get the following fundamental result.

3.1. Theorem. *Let f be a homeomorphism of a compact* ED-*space K into itself. There exists a decomposition $\{B_i \mid i = 0, 1, 2, 3\}$ of K with B_i closed-open such that f is an identity on B_0, and $f[B_j] \cap B_j = \emptyset$ for $j = 1, 2, 3$. In particular, the set of all fixed points of f is closed-open.*

Remark. The set of all p-periodic points of f in Theorem is closed-open because Theorem applies to the p-fold composite of f by itself.

It follows from Theorem 3.1. that a homeomorphism of a compact ED-space into a nowhere dense set of itself has no fixed points. As a consequence we get the following useful result for proving nonhomogeneity of spaces (see [8], [9], [10]).

3.2. Proposition. *Assume that E is a closed nowhere dense subspace of an extremally disconnected compact space K, and assume that E contains a copy of K. There exist $x, y \in E$ such that $hx = y$ for no homeomorphism h of E into itself, in particular E is not homogeneous.*

Proof. Choose any homeomorphism of K into E, pick any y in E, and put $x = gy$.

3.3. The set F of all fixed points of a continuous mapping of K into itself is always closed, however, it need not be open, even if K is a compact ED-space. E.g., consider a constant mapping. For more complicated examples see [11], Examples 1, 2 and 3.

In our particular case we can prove an analogon of 3.1. Let K be a Čech-Stone compactification of a discrete space M; we write $K = \beta M$. If f is a continuous mapping of K into itself such that $f[M] \subset M$, then F is open because F is the closure of $F \cap M$. This follows from Lemma on three sets (see 1.3).

On the other hand it is not difficult to prove that the fixed points are nonexpanding.

3.4. Theorem. *Let x be a fixed point of a continuous mapping f of a space E into itself. It E is an* ED-*space, then each neighborhood of x contains an invariant closed-open neighborhood. If E is a basically disconnected space then each neighborhood of x contains an invariant closed neighborhood.*

3.5. Example in [10] shows that in 2.4, 2.3, 2.2, 2.1 and 1.2 the assumption that the space is an ED-space may not be relaxed to basically disconnected (dual of σ-complete) or to the dual of m-complete.

4. Topological Theory of Types of Ultrafilters

Let βM be a Čech-Stone compactification of a discrete space M (we assume that $M \subset \beta M$, i.e., that M is identically embedded in βM). If $x \in \beta M - M = M^*$ then the intersections of the neighborhoods of x with M form a free ultrafilter on M, and each free ultrafilter on M is of that form. We shall assume that no free ultrafilter on M is an element of M, and we may and shall think of the points of M^* as free ultrafilters on M.

Remark. Let M be a discrete space; then the set $\exp M$ of all subsets of M is the Boolean algebra of all closed-open sets in M, the structure space K of $\exp M$ consists of all ultrafilters in M, and if we assume to each $m \in M$ the ultrafilter on M containing the singleton (x) we get a compactification $f: M \to K$ of M such that the points are just the ultrafilters. Now we replace fm by m, and get the Čech-Stone compactification described above. Of course we may replace fm by m if $m \notin K^*$.

4.1. We denote by N the discrete space of natural numbers, and let identity: $\mathsf{N} \to \beta\mathsf{N}$ be the Čech-Stone compactification such that the elements of $\mathsf{N}^* = \beta\mathsf{N} - \mathsf{N}$ are just the free ultrafilters as described above. It is well-known that (consult any book on topology, e.g. [12]):

(a) *The cardinal of* $\beta\mathsf{N}$ *is* $\exp\exp\aleph_0$.

(b) *If* X *is dense in* N^*, *then the cardinal of* X *ist at least* $\exp\aleph_0$.

4.2. For brevity, if not otherwise stated, then by an ultrafilter we mean a free ultrafilter on a countable set.

4.2.1. Definition of types. *Let F be an ultrafilter on M_i, $i = 1, 2$. We write $F_1 \sim F_2$, and say that F_1 and F_2 are equivalent if there exists a bijective mapping f of M_1 onto M_2 such that $F \in F_1$ if and only if $f[F] \in F_2$. Clearly $\sim$ is an equivalence. Let τ be a fixed single-valued relation which assigns to each ultrafilter F an element τF, called the type of F, such that $\tau F_1 = \tau F_2$ if and only if $F_1 \sim F_2$. We also say that F is of type τF. The set of all types is denoted by* T.

4.2.2. *The cardinal of* T *is* $\exp\exp\aleph_0$.

Proof. The restriction of τ to $\mathsf{N}^* = \beta\mathsf{N} - \mathsf{N}$ is onto T. The cardinal of N^* is $\exp\exp\aleph_0$, and each ultrafilter on N is equivalent to at most $\exp\aleph_0$ ultrafilters because there is $\exp\aleph_0$ permutations of N.

4.2.3. Definition. *Let F be an ultrafilter on M, and let $\{\mathscr{F}_m \mid m \in M\}$ be a family of ultrafilters. The collection of all sets*

$$\sum\{F_m \mid m \in F\},$$

with $F \in \mathscr{F}$, $F_m \in \mathscr{F}_m$, *is an ultrafilter which is called the sum of* $\{\mathscr{F}_m\}$ wrt $\mathscr{F}$, *and designated by*

$$\sum_{\mathscr{F}}\{\mathscr{F}_m \mid m \in M\} .$$

If $\{\mathscr{G}_m \mid m \in M\}$ *is a family of ultrafilters such that* $\mathscr{F}_m \sim \mathscr{G}_m$ *for each* m, *then*

$$\sum_{\mathscr{F}}\{\mathscr{F}_m\} \sim \sum_{\mathscr{F}}\{\mathscr{G}_m\}$$

which enables us to define, in the natural way, the sum of a family $\{t_m\}$ *of types of ultrafilters* wrt *an ultrafilter* $\mathscr{F}$; *notation*: $\sum_{\mathscr{F}}\{t_m\}$.

Remarks. The sum of ultrafilters comes out naturally when considering iterated limits with filters. See [15], Introduction.

4.2.4. *Let* P *be a space, and let* $x \in P$. *If* $X \subset P$ *is a normally embedded discrete countable set such that* $x \in X^*$, *then the intersections of* X *with the neighborhoods of* x *form an ultrafilter on* X *that is called the ultrafilter of* x wrt X *in* P, *and that is designated by* $\mathscr{F}(x, X, P)$. *The type of* $\mathscr{F}(x, X, P)$ *is called the type of* x wrt X *in* P, *and is designated by* $t(x, X, P)$. *The set of all* $t(x, X, P)$ *with* x *and* P *fixed is denoted by* $\mathsf{T}(x, P)$.

Consider $P = \beta\mathsf{N}$. Let $G = \sum_{\mathscr{F}}\{\mathscr{F}_m \mid m \in M\}$. Choose a discrete family $\{x_m \mid m \in M\}$ of points in $\beta\mathsf{N}$ such that

$$t(x_m, \mathsf{N}, \beta\mathsf{N}) = \tau\mathscr{F}_m ,$$

and choose a point x in $\beta\mathsf{N}$ in the closure of the set X of all x_m such that

$$t(x, X, \beta\mathsf{N} = \tau\mathscr{F} .$$

It is easy to see that

$$t(x, \mathsf{N}) = \tau\mathscr{G} .$$

This is the topological description of the sum of ultrafilters. Observe that if $y \in \beta\mathsf{N}$ and if $t(y, \mathsf{N}) = \tau\mathscr{G}$, then there exist $x_m \in \beta\mathsf{N}$ such that the above representation of $\sum_{\mathscr{F}}\{\mathscr{F}_m\}$ holds with $x = y$.

4.2.5. *If* $\mathscr{G} = \sum_{\mathscr{F}} \{\mathscr{F}_m\}$ *then* $\tau\mathscr{G} \neq \tau\mathscr{F}$.

Proof. Let x, x_m have the meaning in 4.2.4. If $\tau\mathscr{G} = \tau\mathscr{F}$ then there would exist a bijective mapping f of N onto X such that $f[\mathscr{F}(x, N)] = \mathscr{F}(x, X)$. Then f extends to a homeomorphism of $\beta\mathsf{N}$ into $\beta\mathsf{N}$; clearly $fx = x$. On the other hand, $f[\beta\mathsf{N}]$

is nowhere dense, and therefore, by 3.1, there is no fixed point. This contradiction proves 4.2.5.

4.2.6. We write $t < t_1$, and say that t_1 is produced by t, if

$$\mathscr{G} = \sum_{\mathscr{F}} \{\mathscr{F}_m\} ,$$

for some $\mathscr{F}_m$, $\mathscr{G}$ with $\tau\mathscr{G} = t_1$, and $\mathscr{F}$ with $\tau\mathscr{F} = t$. In this situation we write $\mathscr{F} < \mathscr{G}$. Clearly $<$ is an order on T. By 4.2.5 we have.

Theorem (Theorem 1 in [6].) *$t < t_1$ for no t in* T.

Thus $<$ is irreflexive. This is the first important property of $<$. The other one says:

4.2.7. Theorem (Theorem C in [7]). *Every type is produced by at most* $\exp \aleph_0$ *types, stated formally,* card $<^{-1} [t] \leqq \exp \aleph_0$ *for each t in* T.

We are able to prove the following important result.

4.2.8. Theorem ([9], Theorems 2, 3). *Let P be a closed subspace of an* ED-*space (or more generally, let P admit an embedding as a closed subspace into a basically disconnected space). For each x in P the set* $\mathsf{T}(x)$ *is linearly ordered, and if $t(x, X) = t(x, Y)$, then $t(x, X) = t(x, X \cap Y)$.*

Proof. All stastements follow from 1.8, see also 1.9.

Remark. The second statement may be formulated as follows: let $\mathfrak{m}(x)$ be the set of germs of countable discrete sets X at x with $x \in X^*$, and let $\langle X \rangle$ be the germ which contains X. Define $g_1 < g_2$ if $X_1 \subset X_2^*$ for some X_i with $g_i = \langle X_i \rangle$. The natural mapping of $\mathfrak{m}(x)$ onto $T(x)$ is one-to-one order-preserving.

4.2.9. *Let $\mathscr{F}_i$ be an ultrafilter on M_i, $i = 1, 2$. We write $\mathscr{F}_1 \to \mathscr{F}_2$ if there exists a mapping f of M_1 into M_2 such that $f^{-1}[F] \in F_1$ for each F in F_2. This relation induces an order $\to$ on T. If $t_1 > t_2$ or $t_1 = t_2$ then $t_1 \to t_2$.*

Clealy $\mathscr{F}_1 \to \mathscr{F}_2$ if and only if there exists a continuous $g : \beta M_1 \to \beta M_2$ such that $g\mathscr{F}_1 = \mathscr{F}_2$, and $g[M_1] \subset M_2$.

4.2.10. Theorem ([15], Proposition 15). *If $t \to t_1 \to t$, then $t = t_1$.*

Proof. Let $\mathscr{F}$ and $\mathscr{F}_1$ be ultrafilters on N, f and f_1 be mappings corresponding to $\mathscr{F} \to \mathscr{F}_1$ and $\mathscr{F}_1 \to \mathscr{F}_2$, respectively. Then $f = f_2 \circ f_1$ is an identity on an element F of $\mathscr{F}$ by Lemma on three sets, and so on. Compare 4.2.10 with Theorem 4.2.6.

4.3. Multiplication of types. *The product* $\mathcal{G} . \mathcal{F}$ *of two ultrafilters is defined to be the sum* $\sum_{\mathcal{G}}\{\mathcal{F}_m \mid m \in M\}$ *where* $M = \bigcup \mathcal{G}$, *and* $\mathcal{F}_m = \mathcal{F}$ *for* $m \in M$. *This induces multiplication of types.*

4.3.1. Theorem. $t < t . t_1 \neq t_1$ *for every* $t, t_1 \in \mathsf{T}$.

The first relation is a particular case of 4.2.6. The proof of the second one goes as follows. Assume that $t . t_1 = t_1$. Then we can find a discrete set X in N^*, and $y \in X^*$, and a continuous mapping f of $\beta\mathsf{N}$ into $\beta\mathsf{N}$ such that $f[\mathsf{N}] \subset \mathsf{N}$, and $fn \neq n$ for n in N, and:

(a) $t(x, N) = t(y, N) = t, t(y, X) = t_1$.

(b) $f[X] = (y)$.

Then $fy = y$ by (b), and this contradicts to Lemma on three sets.

4.3.2. M. Katětov has proved that $t . t_1 \neq t_1 . t$ for some t and t_1, see [15], Theorem 4.7.

4.3.3. *Let* $t \in T$, *and let* A *be the set of all* x *in* $\beta\mathsf{N}$ *of type* t. *By* 4.3.1 *each countable discrete set in* A *is closed.*

I do not know for which t the space A is pseudocompact. It is not if t is a p-type as described in 4.4.1.

For any t the space $\mathsf{N} \cup A$ is pseudocompact (because each subset of N has a cluster point in A), and in addition, every power $(\mathsf{N} \cup A)^{\aleph}$ is pseudocompact. This fact was used in [5] to exhibit an example of a space X such that each finite product X^n is pseudocompact, but $X^{\aleph_0}$ is not, and given a positive integer k, a space Y_k such that Y_k^k is pseudocompact but Y_k^{k+1} is not.

4.4. Examples

4.4.1. Proposition. *The following properties of an ultrafilter* $\mathcal{F}$ *on* N *are equivalent*:

(a) $\mathcal{F}$ *is a P-point of* N^*;

(b) *For each sequence* $\{F_n\}$ *in* $\mathcal{F}$ *there exists an* F *in* $\mathcal{F}$ *such that* $F - F_n$ *is finite for each* n.

(c) *If* f *is any mappoing of* N *into the closed interval of reals then* $f[F]$ *has at most one cluster point for some* F *in* $\mathcal{F}$ (*and hence,* $\{fn \mid n \in F\}$ *converges*).

(d) *If* N_n *is a partition of* N, *then* $\mathsf{N}_n \in \mathcal{F}$ *for some* n *or* $F \cap \mathsf{N}_n$ *is finite for some* F *in* $\mathcal{F}$ *and each* n.

An ultrafilter with properties (a)–(d) is called a P-ultrafilter (or δ-stable or 1-simple by G. Choquet [4]). This terminology carries over to types. The proof of equivalence of (a)–(d) is quite routine. On the other hand, all existence proofs of P-types depend on the Continuum Hypothesis (abbreviated: CH). W. Rudin proved existence of P-points in N^* (hence, P-types) in [17]; his method was refined by G. Choquet in [3] and [4]. The constructions are based on the following simple lemma:

If $\mathscr{F}$ is a countable filter base, then there exists a set G such that $G - F$ is finite for each F in $\mathscr{F}$.

4.4.2. *Under* CH *there exist* exp exp $\aleph_0$ *P-types. In particular,* $\langle \mathsf{T}, < \rangle$ *is not linearly ordered.*

Proof. Rudin [11] or Choquet [4].

4.4.3. Every P-type is produced by no type, in particular, P-types are minimal elements of $\langle \mathsf{T}, < \rangle$. It is not known whether there exists a minimal element which is not any P-type.

4.4.4. *An ultrafilter $\mathscr{F}$ on M is said to be rare if for each partition $\{M_n\}$ of M with M_n finite, there exists an F in $\mathscr{F}$ with $F \cap M_n$ at most one-point for each n.*

An ultrafilter $\mathscr{F}$ on N is said to be rapid if for each increasing sequence $\{k_n\}$ of integers there exists an element F of $\mathscr{F}$ such that the n-th element of F is greater than k_n.

It is easy to verify that every rare ultrafilter is rapid.

4.4.5. Theorem. *The following properties of an ultrafilter $\mathscr{F}$ on M are equivalent:*

(a) *$\mathscr{F}$ is a rare P-ultrafilter.*

(b) *For each partition $\{M_n\}$ of M, either $M_n \in \mathscr{F}$ for some n, or $F \cap M_n$ is at most one-point for some F in $\mathscr{F}$ and each n.*

(c) *$\mathscr{F}$ is minimal in $\langle \mathsf{T}, \rightarrow \rangle$.*

An $\mathscr{F}$ with properties (a)–(e) is called a $\rightarrow$ – minimal (or absolute by G. Choquet). The proof is easy.

4.4.6. G. Choquet [4] has shown, of course under CH, that the concepts of P-ultrafilter and rare ultrafilter are "completely" independent. For our purposes, it is important that there exists a P-type which is not any $\rightarrow$ – minimal type.

5. Non-homogeneity of Compact ED-spaces

We say that a space P is homogeneous if for each x and y in P there exists a homeomorphism h of P onto P with $hx = y$. The following is evident:

5.1. Proposition. *If a space P is homogeneous then $\mathsf{T}(x, P) = \mathsf{T}(y, P)$ for each x and y in P. If a space P is homogeneous and if P contains a copy of $\beta\mathsf{N} - \mathsf{N}$, then $\mathsf{T}(x, P) = \mathsf{T}$ for each x in P.*

5.2. Theorem. *Assume that K is an infinite compact space that is an* ED-*space (or more generally, that admits an embedding into a basically disconnected space). If K is homogeneous then*

(a) $\mathsf{T}(x, K) = \mathsf{T}$ *for each x in K,*

(b) T *is linearly ordered,*

(c) *There is no cardinal between* $\exp \aleph_0$ *and* $\exp \exp \aleph_0$.

Proof. Assertion (a) follows from 4.2.8. Assertion (b) follows from (a) by 5.1. Statement (c) follows from (a), 4.2.1, and from the following observation:

Assume that $t = t(x, X)$. Consider the closure L of X in K; L is a copy of $\beta\mathsf{N}$, and $t' \in \mathsf{T}$, $t' < t$ if and only if $t = t(x, Y)$ for some $Y \subset X^*$. Thus the set

$$\mathscr{E}\{t' \mid t' = \text{or } t' < t\} = \mathsf{T}(x, L)\,.$$

The cardinal of $\mathsf{T}(x, L)$ is at most $\exp \aleph_0$ by 4.2.

5.3. Theorem ([9], Theorem 1). *If there is no cardinal between $\aleph_0$ and* $\exp \aleph_0$, *or if there is a cardinal between* $\exp \aleph_0$ *and* $\exp \exp \aleph_0$, *then there exists no infinite compact space that admits an embedding into an* ED-*space (or more generally, basically disconnected space).*

5.4. I cannot prove that the assumptions in Theorem 5.3 on the set theory used may be omitted. However for many spaces "absolute" non-homogeneity follows from preceding results. E.g., Proposition 3.2. gives absolute nonhomogeneity of closed subspace of $\beta\mathsf{N} - \mathsf{N}$, some of $\beta M - M$ in general, see [9, closing Remarks].

6. Non-homogeneity of $\beta P - P$

6.1. Theorem ([8], Theorem). *If P is not pseudocompact then $\beta P - P$ is not homogeneous (without any assumption on the set theory).*

Under the additional assumption that P is locally compact and CH holds W. Rudin formulated and proved Theorem 6.1 in [17]. T. Isiwata proved Rudin's Theorem without assuming P locally compact.

The proof of 6.1 is based on the following Lemma and the theory of types.

6.2. Lemma (Lemma in [9]). *Let X be a completely normally embedded countable subset of a space P. Let $Y \subset \beta P$ be a countable set which is semi-separated to X. Then X and Y are functionally separated in βP (and hence: $\mathrm{cl}\, X \cap \mathrm{cl}\, Y = \emptyset$).*

6.3. Proof of 6.1. Since P is not pseudocompact there exists a completely normally embedded discrete countable infinite set X in P. Take any point x in $\mathrm{cl}\, X - X$. It follows from 6.2 that $\mathsf{T}(x, \beta P - P) \neq \mathsf{T}$. Since $\beta P - P$ contains a copy of $\beta \mathsf{N} - \mathsf{N}$, namely $\mathrm{cl}\, X - X$, $\beta P - P$ is not homogeneous by 5.1.

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