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Uniformities and embeddings

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UNIFORMITIES AND EMBEDDINGS[1])

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1. Introduction

In this report we are interested in considering the concept of a uniformity defined on a subspace of a topological space. The questions that we will consider are similar to those demanded when one considers the analogous concept of extending continuous real valued functions on a subspace. Since every uniformity given by entourages determines a collection of pseudometrics, and since conversely every collection of pseudometrics on a space determines an entourage uniformity, we will also consider the corresponding notions for extensions of pseudometrics defined on the subspace. Our results will give us new characterizations of C-embedding and C^*-embedding. In particular we can give a new characterization for collectionwise normality, the Čech-Stone compactification βX and the Hewitt real-compactification υX of a topological space X.

Our ideas have their foundation in the early works of Hausdorff, Bing, and Arens. In 1930, F. Hausdorff [12] showed that a continuous metric defined on a closed subset S of a metric space X can be extended to a continuous metric on X. About twenty years ago, R. H. Bing [6] and R. Arens [4] rediscovered this result independently. Arens extended the results further for normal spaces. Other authors [1], [19], and [5] have considered related notions such as considering subspaces S for which every continuous pseudometric defined on S has a continuous pseudometric extension to X. Such subspaces S are said to be *P-embedded* in X. It is known that P-embedded subspaces are C-embedded but not conversely. However if S is a pseudo-compact, C^*-embedded subset of a completely regular space X then S is P-embedded.

Since a collection of pseudometrics on a space generates a uniformity, from the above considerations other questions come. It is well known that any compactification γX of a space X has an unique *admissible* uniform structure, $\mathcal{U}^*$, whereas X may have many distinct admissible uniform structures. Therefore there is a unique admissible structure $\mathcal{U}$ on X such that $\mathcal{U}$ is the relative structure on X obtained from $\mathcal{U}^*$. Deleting the word "admissible", we call $\mathcal{U}^*$ an *extension* of $\mathcal{U}$ to γX. Immediately

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the following question arises. If S is a subspace of a completely regular space X then what conditions placed on S and X will guarantee the existence of admissible extensions to X of various admissible uniformities on S? Such a question has been studied in [10]. Here we will consider the same question with the relaxation of the condition of admissibility of the extension uniformity to the condition of continuity of the extension uniformity. We will show that S is P-embedded in X if and only if very admissible uniformity on S has a continuous extension to X; S is C (C^*)-embedded in X if and only if every admissible uniformity on S generated by a collection of continuous (bounded continuous) real valued functions has a continuous extension. Immediately these then give new characterizations of the Čech-Stone compactification, the Hewitt real-compactification, normality and collectionwise normality.

2. Definitions and Terminology

The notions and terminology used in this paper, with one exception, is that of [11]. The exception is that of the definition of a uniform space where entourages are used as in [15]. *All topological spaces are assumed to be Tichonov spaces* (i.e. *completely regular and* T_1).

As mentioned above any collection $\mathscr{D}$ of pseudometrics on a topological space $(X, \mathscr{T})$ gives rise to a uniformity. The sets $U(d, \varepsilon) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$ for all d in $\mathscr{D}$ and all $\varepsilon > 0$ form a subbase for some uniformity. A subcollection $\mathscr{P}$ of subsets of $X \times X$ *generates* a uniformity $\mathscr{U}$ on X if $\mathscr{P}$ is a subbase for $\mathscr{U}$. To every real valued function f on X is associated a pseudometric ψ_f on X defined by

$$\psi_f(x, y) = |f(x) - f(y)| \quad (x, y \in X) .$$

A family of functions $\mathscr{G}$ *generates* a uniformity $\mathscr{U}$ if $\{U(\psi_f, \varepsilon) : f \in \mathscr{G} \text{ and } \varepsilon > 0\}$ generates $\mathscr{U}$. The pseudometric topology associated with a pseudometric d is denoted by $\mathscr{T}_d$. A pseudometric d is said to be $\aleph_0$-*separable* in case the pseudometric space $(X, \mathscr{T}_d)$ has a countable dense subset. The pseudometric is *totally bounded* if for every $\varepsilon > 0$ there is a finite subset F of X such that X is the union of the d-spheres of radius ε centered at the points of F.

A *precompact uniformity* is a uniformity $\mathscr{U}$ on X that is generated by a collection of bounded continuous real valued functions on X. If the uniformity is generated by a collection of continuous real valued functions then the uniformity is said to be *prerealcompact.*

Every uniformity $\mathscr{U}$ on a non-empty set X yields a unique topology $\mathscr{T}(\mathscr{U})$. This topology is obtained by taking as a base for the neigborhoods the collection of sets $U[x]$ for all U in $\mathscr{U}$ and all x in X. If $\mathscr{T}(\mathscr{U})$ is a subcollection of the original topology $\mathscr{T}$ on X then we say that $\mathscr{U}$ is a *continuous uniformity*. If $\mathscr{T}(\mathscr{U})$ agrees with $\mathscr{T}$ then $\mathscr{U}$ is called an *admissible uniformity*. The admissible uniformity on X generated by the collection of all continuous pseudometrics (respectively, all continuous real valued functions, all bounded continuous real valued functions) on X is denoted by $\mathscr{U}_0(X)$ (respectively, $\mathscr{C}(X)$, $\mathscr{C}^*(X)$).

If S is a subset of X and if $\mathscr{G}$ is a collection of subsets of X, then by $\mathscr{G} \mid S$ is meant the collection $\{G \cap S : G \in \mathscr{G}\}$. A uniformity $\mathscr{U}^*$ on X is an *extension* of a uniformity $\mathscr{U}$ on S in case $\mathscr{U}^* \mid S \times S = \mathscr{U}$. The subset S is *uniformly embedded in X* in case every admissible uniformity $\mathscr{U}$ on S has an extension that is a continuous uniformity on X. It is said to be *prerealcompact uniformly embedded in X* in case every prerealcompact admissible uniformity $\mathscr{U}$ on X has an extension that is a continuous uniformity on X. The subset S is *precompact uniformly embedded in X* if every precompact admissible uniformity $\mathscr{U}$ on X can be extended to a continuous uniformity on X. The subset S is *P-embedded* in X if every continuous pseudometric on S has a continuous pseudometric extension to X.

For the sake of completeness we also include the following definitions. The family $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ of subsets of a topological space X is said to be *discrete* at a point $x \in X$ if there is a neighborhood G of x that meets at most one member of $\mathscr{U}$. The family $\mathscr{U}$ is discrete if $\mathscr{U}$ is discrete at each point $x \in X$. The space X is a *collectionwise normal* space if for every discrete family $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ of closed subsets of X there is a family $(G_\alpha)_{\alpha \in I}$ of mutually disjoint open subsets of X such that $U_\alpha \subset G_\alpha$ for each $\alpha \in I$. (This definition is due to R. H. Bing [2].)

If $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is a cover of X and if $\mathscr{V} = (V_\beta)_{\beta \in J}$ is a cover of S, then $\mathscr{U}$ is an *extension* of $\mathscr{V}$ if $I = J$ and if $U_\alpha \cap S = V_\alpha$ for all $\alpha \in I$. A sequence $(\mathscr{U}_n)_{n \in \mathbf{N}}$ of covers of a set X is said to be a *normal sequence* in case $\mathscr{U}_{n+1}$ is a star refinement of $\mathscr{U}_n$. A cover $\mathscr{U}$ of X is said to be a *normal cover* in case there exists a normal sequence $(\mathscr{U}_n)_{n \in \mathbf{N}}$ of open covers of X such that $\mathscr{U}_1$ is a refinement of $\mathscr{U}$. (This definition is due to Tukey [11].) If $(\mathscr{U}_n)_{n \in \mathbf{N}}$ is a normal sequence of open covers of a space X and if d is a pseudometric on X, then d is *associated with* $(\mathscr{U}_n)_{n \in \mathbf{N}}$ if the following three conditions are satisfied:

(1) d is bounded by the constant function $\mathbf{1}$.

(2) If $k \in \mathbf{N}$ and if $d(x, y) < 2^{-(k+1)}$, then $x \in \operatorname{st}(y, \mathscr{U}_k)$ (the star of y with respect to $\mathscr{U}_k$).

(3) If $k \in \mathbf{N}$ and if $x \in \operatorname{st}(y, \mathscr{U}_k)$, then $d(x, y) < 2^{-(k-3)}$.

For any real valued continuous function defined on X, the set, $Z(f)$, of all points x in X for which $f(x) \neq 0$ is the *zero-set of f*. The complement of $Z(f)$ is called the *cozero-set* of f. The family $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is a *cozero-set cover* of X if $\mathscr{U}$ is a cover of X and if each U_α is a cozero-set. A *zero-set cover* is defined in an analogous manner.

If $(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ is a finite sequence of covers of a set X and if $\mathscr{A}_i = (A_i(\alpha))_{\alpha \in J_i}$ for each $i = 1, \ldots, n$, then by $\bigwedge_{i=1}^n \mathscr{A}_i$ we mean the family

$$(A_1(\alpha_1) \cap \ldots \cap A_n(\alpha_n))_{(\alpha_1, \ldots, \alpha_n) \in J_1 \times \ldots \times J_n}.$$

In the definition of a prerealcompact uniformly embedded subset and a precompact uniformly embedded subset we required only that the extension be a continuous uniformity. Our first theorem shows that these extensions can be taken to be prerealcompact in the case of a prerealcompact uniformly embedded subset and precompact in the case of a precompact uniformly embedded subset.

2.1. Theorem. *If S is a subset of a topological space $(X, \mathscr{T})$ then for the following conditions, (1) is equivalent to (2) and (3) is equivalent to (4).*

(1) *S is prerealcompact uniformly embedded in X.*

(2) *Every prerealcompact admissible uniformity on S has a continuous extension that is prerealcompact.*

(3) *S is precompact uniformly embedded in X.*

(4) *Every precompact admissible uniformity on S has a continuous extension that is precompact.*

Proof. Trivially (1) is implied by (2) and (3) is implied by (4). To see that (1) implies (2), let $\mathscr{U}$ be an admissible prerealcompact uniformity on S. By (1), there exists a continuous uniformity $\mathscr{U}^*$ on X that is an extension of $\mathscr{U}$. Let $\mathscr{A}$ be the collection of continuous real valued functions on S that generate $\mathscr{U}$, let $\mathscr{B}$ be the collection of $\mathscr{T}(\mathscr{U}^*)$-continuous real valued functions f on X such that $f \mid S \in \mathscr{A}$, and let $\mathscr{V}^*$ be the uniformity on X generated by $\mathscr{B}$. Now $(S, \mathscr{U})$ is a uniform subspace of the uniform space $(X, \mathscr{U}^*)$ and in [10, Theorem 1] it was shown that every uniformly continuous real valued function on $(S, \mathscr{U})$ has a continuous real valued extension to $(X, \mathscr{U}^*)$. Thus, since every function in $\mathscr{A}$ is uniformly continuous with respect to $\mathscr{U}$, it follows that its continuous real valued extension to $(X, \mathscr{U}^*)$ is a member of $\mathscr{B}$ and hence is uniformly continuous with respect to $\mathscr{V}^*$. Thus the uniformity $\mathscr{U}$ is a subcollection of $\mathscr{V}^* \mid S \times S$. On the other hand it is clear that $\mathscr{V}^* \mid S \times S \subset \mathscr{U}$ and thus $\mathscr{V}^*$ is an extension of $\mathscr{U}$. Since $\mathscr{U}^*$ is a continuous uniformity on X and each function in $\mathscr{B}$ is $\mathscr{T}(\mathscr{U}^*)$-continuous, it follows that they are also $\mathscr{T}$-continuous. Hence $\mathscr{V}^*$ is a continuous uniformity and it is also prerealcompact. Hence (1) is equivalent to (2).

For the case that (3) implies (4) the following adjustments in the above proof may be made. For $\mathscr{U}$ an admissible precompact uniformity on S, let $\mathscr{A}$ be the collection of bounded continuous real valued functions on S that generate $\mathscr{U}$, let $\mathscr{B}$ be the collection of bounded $\mathscr{T}(\mathscr{U}^*)$-continuous real valued functions on X whose traces belong to $\mathscr{A}$ and let $\mathscr{V}^*$ be as above. In [13, Theorem 3] it was shown that every bounded uniformly continuous real valued function on $(S, \mathscr{U})$ has a bounded uniformly continuous real valued extension to $(X, \mathscr{U}^*)$. Using this stronger result the proof proceeds exactly as above.

2.2. Remark. In the last proof the uniformity $\mathscr{V}^*$ may be chosen to be a subcollection of $\mathscr{U}^*$. This can be done by choosing for $\mathscr{B}$ only those bounded $\mathscr{U}^*$-uniformly continuous functions on X whose traces belong to $\mathscr{A}$.

The following result will be needed for the proof of the main theorem. The universal uniformity on a topological space is the largest continuous uniformity on the space.

2.3. Lemma. *If $\mathscr{V}$ is a continuous uniformity on $(X, \mathscr{T})$ then $\mathscr{V} \subset \mathscr{U}_0(X)$*

3. *P*-Embedding

We can now state and prove our main results for *P*-embedding.

3.1. Theorem. *Let S be a subset of a topological space $(X, \mathscr{T})$. Then S is P-embedded in X if and only if S is uniformly embedded in X.*

Proof. To prove sufficiency, it is necessary to recall that S is P-embedded in X if and only if $\mathscr{U}_0(S) = \mathscr{U}_0(X) \mid S \times S$ (see [10, Theorem 7.5]) and that for any subspace S of X, $\mathscr{U}_0(X) \mid S \times S$ is always contained in $\mathscr{U}_0(S)$. Since $\mathscr{U}_0(S)$ is an admissible uniformity, S uniformly embedded in X implies that there is a continuous uniformity $\mathscr{V}$ on X such that $\mathscr{V} \mid S \times S = \mathscr{U}_0(S)$. For any $U \in \mathscr{U}_0(S)$ there is a $V \in \mathscr{V}$ so that $V \cap (S \times S) = U$. The lemma above then yields that U is a member of $\mathscr{U}_0(X) \mid S \times S$ and it follows that S is P-embedded in X.

For the necessity of the condition, let $\mathscr{U}$ be an admissible uniformity on S, let $\mathscr{P}$ be the set of all continuous pseudometrics on $(X, \mathscr{T})$ satisfying:

if $d \in \mathscr{P}$ then $d \mid S \times S$ is uniformly continuous on S relative to $\mathscr{U}$,

and let $\mathscr{U}^*$ be the uniformity on X generated by $\mathscr{P}$. The uniformity $\mathscr{U}^*$ is a continuous extension of $\mathscr{U}$; in fact the subbasic elements of $\mathscr{T}(\mathscr{U}^*)$ are the d-spheres of radius ε about each $x \in X$ for $d \in \mathscr{P}$ and $\varepsilon > 0$. The continuity of each $d \in \mathscr{P}$ relative to $\mathscr{T}$ implies that $\mathscr{T}_d \subset \mathscr{T}$ and hence $\mathscr{T}(\mathscr{U}^*) \subset \mathscr{T}$. Moreover, $\mathscr{U}^* \mid S \times S = \mathscr{U}$. If U is any member of $\mathscr{U}$ then there is a continuous pseudometric d on S and an $\varepsilon > 0$ so that

$$W = \{(x, y) \in S \times S : d(x, y) < \varepsilon\} \subset U .$$

Since S is P-embedded in X, d has a continuous pseudometric extension d^* in $\mathscr{P}$. Let

$$W^* = \{(x, y) \in X \times X : d^*(x, y) < \varepsilon\} .$$

Then $W^* \in \mathscr{U}^*$ and

$$W^* \cap (S \times S) = W \subset U \in \mathscr{U}^* \mid S \times S .$$

Hence $\mathscr{U} \subset \mathscr{U}^* \mid S \times S$. Conversely, if $U \in \mathscr{U}^* \mid S \times S$, then there is $U^* \in \mathscr{U}^*$ such that $U^* \cap (S \times S) = U$. Hence there is a $d \in \mathscr{P}$ and $\varepsilon > 0$ for which

$$V = \{(x, y) \in X \times X : d(x, y) < \varepsilon\} \subset U^* .$$

Thus $V \cap (S \times S) \subset U^* \cap (S \times S)$ and since $d \mid S \times S$ is uniformly continuous on S relative to $\mathscr{U}$, $V \cap S \times S \in \mathscr{U}$ and hence $U \in \mathscr{U}$. This completes the proof.

3.2. Corollary. *A completely regular space $(X, \mathscr{T})$ is collectionwise normal if and only if every closed subset is uniformly embedded in X.*

Proof. This follows from the Theorem and [19, Theorem 5.2].

3.3. Corollary. *Let f be a closed continuous function from a topological space X onto a topological space Y. Then X is paracompact if and only if Y is paracompact and $f^{-1}(y)$ is paracompact and uniformly embedded in X for each y in Y.*

Proof. This follows from the Theorem and [11, Theorem 1.]

4. C-Embedding

To prove our main result concerning C-embedding we will need the following results. The following two results give us the relationship between the topological structure induced by a pseudometric and the given topology on the space.

4.1. Proposition. *If d is a pseudometric on $(X, \mathscr{T})$ then d is continuous if and only if $\mathscr{T}_d \subset \mathscr{T}$.*

4.2. Proposition. *If d is a continuous pseudometric on $(X, \mathscr{T})$ and G is an open subset of X relative to $\mathscr{T}_d$, then G is a cozero-set relative to $\mathscr{T}$.*

4.3. Theorem. *If $(\mathscr{U}_n)_{n \in \mathbf{N}}$ is a normal sequence of open covers of $(X, \mathscr{T})$ then there is a continuous pseudometric on X that is associated with $(\mathscr{U}_n)_{n \in \mathbf{N}}$.*

This theorem was first shown by Tukey ([22], Theorem 7.1) using results due to A. H. Frink [9]. The following is a useful fact about open covers and is shown in ([19]; Proposition 2.5).

4.4. Proposition. *Let $\mathscr{U} = (\mathscr{U}_\alpha)_{\alpha \in I}$ be an open cover of the subset S of $(X, \mathscr{T})$ and let $\mathscr{B}$ be a normal open cover of X such that $\mathscr{B} \mid S$ refines $\mathscr{U}$. Then there is a normal locally finite cozero-set cover $\mathscr{W} = (W_\alpha)_{\alpha \in I}$ of X such that $W_\alpha \cap S \subset U_\alpha$ for each $\alpha \in I$.*

From the proof of Theorem 2 in [8] we obtain

4.5. Theorem. *Every countable cozero-set cover of $(X, \mathscr{T})$ has a countable locally finite cozero-set refinement.*

4.6. Corollary. *Every countable cozero-set cover of a topological space is normal.*

Proof. This follows from 4.5 and the fact that in ([18], Theorem 1.2), Morita has shown that for a cover having a locally finite cozero-set refinement is equivalent to its being normal.

4.7. Theorem. *Every countable cozero-set cover of $(X, \mathscr{T})$ has a countable star-finite (normal) cozero-set refinement.*

Proof. This is obtained from 4.5 and 4.6 and a modification of ([16], Theorem 3).

We can now state our main result in this section.

4.8. Theorem. *If S is a subset of $(X, \mathscr{T})$, then the following are equivalent.*

(1) *S is C-embedded in X.*

(2) *Every countable star-finite cozero-set cover of S has a refinement that can be extended to a countable cozero-set cover of X.*

(3) *Every countable locally finite cozero-set cover of S has a refinement that can be extended to a normal open cover of X.*

(4) *Every countable normal open cover of S has a refinement that can be extended to a countable locally finite cozero-set cover of X.*

(5) *Every $\aleph_0$-separable continuous pseudometric on S can be extended to a continuous pseudometric on X.*

(6) *Every admissible uniformity on S generated by a collection of $\aleph_0$-separable continuous pseudometrics has a continuous extension to X.*

(7) *S is prerealcompact uniformly embedded in X.*

Proof. (1) implies (2). Let $\mathscr{U} = (U_i)_{i \in \mathbf{N}}$ be a countable star-finite cozero-set cover of S. Thus for each $i \in \mathbf{N}$, there is a $f_i \in C(S)$ such that $f_i(x) \neq 0$ if and only if $x \in U_i$. By hypothesis S is C-embedded in X, so for each $i \in \mathbf{N}$, there exists $f_i^* \in C(X)$ such that $f_i^* \mid S = f_i$. Let $V_i = \{x \in X : f_i^*(x) \neq 0\}$. Since S is C-embedded in X and since $X \setminus (\bigcup_{i \in \mathbf{N}} V_i)$ is a zero-set in X that is disjoint from S, there is a cozero-set V such that $X \setminus (\bigcup_{i \in \mathbf{N}} V_i) \subset V$ and $V \cap S = \emptyset$. Let $W_1 = V \cup V_1$ and $W_i = V_i$ for $i = 2, 3, \ldots$ Since $(W_i)_{i \in \mathbf{N}}$ is a countable cozero-set cover of X that extends $\mathscr{U}$, we have shown (2).

(2) implies (3). Let $\mathscr{U}$ be a countable locally finite cozero-set cover of S. By 4.7, $\mathscr{U}$ has a countable star-finite (normal) cozero-set refinement $\mathscr{B}$. By (2), $\mathscr{B}$ has a refi-

nement that can be extended to a countable cozero-set cover $\mathscr{A}$ of X. By 4.6 $\mathscr{A}$ is normal and hence (3) holds.

(3) implies (4). If $\mathscr{U} = (U_i)_{i\in\mathbf{N}}$ is a countable normal open cover of S, then Morita in ([18], Theorem 1.2) has shown that $\mathscr{U}$ has a countable locally finite cozero-set refinement. By 4.6 this refinement is normal; hence by (3) it has a refinement that can be extended to a normal open cover $\mathscr{B}$ of X. By 4.4 there is a countable locally finite (normal) cozero-set cover $\mathscr{W} = (W_i)_{i\in\mathbf{N}}$ of X such that $W_i \cap S \subset U_i$ for all $i \in \mathbf{N}$. Thus (4) is shown.

(4) implies (5). Let d be an $\aleph_0$-separable continuous pseudometric on S. For each $m \in \mathbf{N}$ let

$$\mathscr{G}_m = (S_d(x, 2^{-(m+3)}))_{x\in S}\,.$$

Let $m \in N$. Since (S, d) is a $\aleph_0$-separable pseudometric space it is Lindelöf. Hence there is a countable locally finite open cover $\mathscr{A}^m$ of S that refines $\mathscr{G}_m$. Moreover $\mathscr{A}^m$ is normal relative to $\mathscr{T}_d$ since open sets are cozero-sets and Morita has shown that locally finite cozero-set covers are normal. Thus by (4) and (4.4) there is a refinement of $\mathscr{A}^m$ that extends to countable locally finite (normal) cozero-set cover $\mathscr{B}^m$ of X. Then there is a normal sequence $(\mathscr{B}_i^m)_{i\in\mathbf{N}}$ of open covers of X such that $\mathscr{B}_1$ refines $\mathscr{B}^m$ and such that for each $i \in \mathbf{N}$, $\mathscr{B}_i^m$ is countable.

For all i, $m \in \mathbf{N}$, let

$$\mathscr{U}^m = \bigwedge_{j=1}^{m} \mathscr{B}^j\,, \quad \mathscr{U}_i^m = \bigwedge_{j=1}^{m} \mathscr{B}_i^j\,.$$

Then for all i, $m \in \mathbf{N}$ one verifies that

(i) $\mathscr{U}^m$ and $\mathscr{U}_i^m$ are countable open covers of X.

(ii) $\mathscr{U}_{i+1}^m$ star refines $\mathscr{U}_i^m$ and $\mathscr{U}_1^m$ refines $\mathscr{U}^m$,

(iii) $\mathscr{U}_i^{m+1}$ refines $\mathscr{U}_i^m$ and $\mathscr{U}^{m+1}$ refines $\mathscr{U}^m$, and

(iv) $\mathscr{U}^m \mid S$ refines $\mathscr{G}_m$.

Again consider any $m \in \mathbf{N}$. It follows from (i) and (ii) that $(\mathscr{U}_i^m)_{i\in\mathbf{N}}$ is a normal sequence of open covers of X. Then by Theorem 4.3 there is a continuous pseudometric r_m on X that is associated with $(\mathscr{U}_i^m)_{i\in\mathbf{N}}$ and r_m is $\aleph_0$-separable. By (ii) and (iv) we have

(a) If $x, y \in S$ and if $r_m(x, y) < 2^{-3}$ then $d(x, y) < 2^{-(m+2)}$. Define $r : X \times X \to \mathbf{R}^+$ by $r(x, y) = \sum_{m\in N} 2^{-m} r_m(x, y)$. One can verify that r is a continuous $\aleph_0$-separable pseudometric on X.

From (a) it follows that

(b) If $x, y \in S$, if $i > 3$ and if $r(x, y) < 2^{-i}$, then $d(x, y) < 2^{-(i-1)}$.

Define a relation R on X as follows:

$$xRy \quad \text{if} \quad r(x, y) = 0 \quad (x, y \in X).$$

Observe that R is an equivalence relation on X and let $X^* = X/R$ be the quotient space of X modulo R with $\tau : X \to X^*$ the quotient map. Defining r^* by

$$r^*(\tau(x), \tau(y)) = r(x, y) \quad (x, y \in X)$$

one shows that (X^*, r^*) is a metric space, $\mathscr{T}_{r^*}$ is the quotient topology on X^*, and τ is an isometry. Since X^* is a continuous image of X (i.e. with respect to $\mathscr{T}_r$ and $\mathscr{T}_{r^*}$) and since X has a countable dense subset A, it follows that X^* also has a countable dense subset (namely $\tau(A)$).

Let $S^* = \tau(S)$. By (b) it follows that we can define a map $d^* : S^* \times S^* \to \mathbf{R}^+$ as:

$$d^*(\tau(a), \tau(b)) = d(a, b) \quad (a, b \in S).$$

Again one can verify that d^* is a pseudometric on S^*.

Let $\mathscr{D}^*$ be the metric uniform structure on S^* whose base consists of the single metric $r^* \mid S^* \times S^*$ ([11], 15.3). By (b) it follows that $d^* \in \mathscr{D}^*$ and therefore d^* is a uniformly continuous function from $S^* \times S^*$ into $\mathbf{R}^+$ ([11], 15N.1). Moreover, by ([11], 15.11), d^* can be extended to a uniformly continuous function d^* from $cS^* \times cS^*$ into $\mathbf{R}^+$. One easily verifies that d^* is a pseudometric on cS^*.

Appealing to the theorem ([3], Theorem 3.4) stated in the beginning of this report, it follows that d^* can be extended to a continuous pseudometric e on X^*. Define $\bar{d}$ on $X \times X$ by $\bar{d} = e \circ (\tau \times \tau)$. Since τ is continuous relative to $\mathscr{T}$, $\bar{d}$ is a continuous pseudometric on X (4.1). Also if $x, y \in S$ then

$$\bar{d}(x, y) = e(\tau(x), \tau(y)) = d^*(\tau(x), \tau(y)) = d(x, y).$$

Therefore $\bar{d} \mid S \times S = d$. Since $\bar{d}$ is continuous relative to $\mathscr{T}_r$ and since r is $\aleph_0$-separable then $\bar{d}$ is $\aleph_0$-separable. Hence (5) holds.

(5) implies (6). Let $\mathscr{U}$ be an admissible uniformity on S generated by a collection $\mathscr{P}$ of $\aleph_0$-separable continuous pseudometrics on S and let $\mathscr{U}^*$ be the uniformity on X generated by the collection $\mathscr{P}^*$ of all continuous pseudometric extensions of members of $\mathscr{P}$. From (5) it follows that $\mathscr{U}^*$ is an extension of $\mathscr{U}$. Moreover, the topology $\mathscr{T}(\mathscr{U}^*)$ is the union of the pseudometric topologies $\mathscr{T}_{d^*}$ and since each d^* is a continuous pseudometric, it follows that $\mathscr{U}^*$ is a continuous uniformity on X.

(6) implies (7). Let $\mathscr{U}$ be an admissible uniformity on S generated by a collection $\mathscr{A}$ of continuous real valued functions on S. Since the usual metric, e, on the real numbers is $\aleph_0$-separable, for each $f \in \mathscr{A}$, the continuous pseudometric $e \circ (f \times f)$ is $\aleph_0$-separable. Hence $\mathscr{U}$ is generated by a collection of $\aleph_0$-separable continuous pseudometrics and (6) applies to give the desired result.

(7) implies (1). Suppose that every admissible uniformity on S generated by a collection of continuous real valued functions has a continuous extension. In particular, since $\mathscr{C}(S)$ is an admissible uniformity, there is a continuous uniformity $\mathscr{U}$ on X such that $\mathscr{U} \mid S \times S = \mathscr{C}(S)$. Hence $(S, \mathscr{C}(S))$ is a uniform subspace of $(X, \mathscr{U})$. As referred to in the proof of Theorem 2.1, every uniformly continuous real valued function on $(S, \mathscr{C}(S))$ has a continuous real valued extension to $(X, \mathscr{U})$. Any continuous real valued function f on S is uniformly continuous with respect to $\mathscr{C}(S)$ and hence has a $\mathscr{T}(\mathscr{U})$-continuous real valued extension f^* to X. Since $\mathscr{U}$ is a continuous uniformity, f^* is also $\mathscr{T}$-continuous and it follows that S is C-embedded in X.

4.9. Corollary. *The Hewitt realcompactification vX of X is that unique realcompact Hausdorff space containing X densely such that every admissible uniformity on X generated by a collection of continuous real valued functions has a continuous extension.*

5. C^*-Embedding

To study C^*-embedding in the same fashion as C-embedding the following adaptations are needed.

5.1. Proposition. *Let $(\mathscr{V}_n)_{n \in \mathbf{N}}$ be a normal sequence of finite open covers of $(X, \mathscr{T})$. If d is a continuous pseudometric on X associated with $(\mathscr{V}_n)_{n \in \mathbf{N}}$, then d is totally bounded.*

5.2. Lemma. *If $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is a finite normal open cover of the space $(X, \mathscr{T})$, then there is a normal sequence $(\mathscr{V}_n)_{n \in \mathbf{N}}$ of open covers of X such that $\mathscr{V}_1$ refines $\mathscr{U}$ and such that $\mathscr{V}_n$ is finite for each $n \in \mathbf{N}$.*

Proof. By hypothesis, there exists a normal sequence $(\mathscr{U}_n)_{n \in \mathbf{N}}$ of open covers of X such that $\mathscr{U}_1$ refines $\mathscr{U}$. Let d be a continuous pseudometric that is associated with $(\mathscr{U}_n)_{n \in \mathbf{N}}$ (4.3). For each $\alpha \in I$, let $W_\alpha = \bigcup\{S_d(x, 2^{-3}) : S_d(x, 2^{-3}) \subset U_\alpha\}$. Then $\mathscr{W} = (W_\alpha)_{\alpha \in I}$ is a finite normal open cover of X relative to $\mathscr{T}_d$ (the topology on X determined by d) such that $W_\alpha \subset U_\alpha$ for each $\alpha \in I$. A repeated application of ([17], Theorem 1.2), and the observation that the covers constructed therein are finite, gives us a normal sequence $(\mathscr{V}_n)_{n \in \mathbf{N}}$ of open covers of X, relative to $\mathscr{T}_d$, such that $\mathscr{V}_1$ refines $\mathscr{W}$ and such that, for each $n \in \mathbf{N}$, $\mathscr{V}_n$ is finite. Since $\mathscr{T}_d \subset \mathscr{T}$ and since $\mathscr{W}$ refines $\mathscr{U}$, the result now follows.

Now our main result in this section is the following.

5.3. Theorem. *If S is a subspace of a topological space $(X, \mathscr{T})$, then the following are equivalent:*

(1) *S is C^*-embedded in X.*

(2) *Every finite (normal) cozero-set cover of S has a refinement that can be extended to a normal open cover of X.*

(3) *Every finite normal open cover of S has a refinement that can be extended to a finite (normal) cozero-set cover of X.*

(4) *Every totally bounded continuous pseudometric on S can be extended to a totally bounded continuous pseudometric on X. (We will say such an S is T-embedded in X).*

(5) *Every totally bounded continuous pseudometric on S can be extended to a continuous pseudometric on X.*

(6) *Every admissible uniformity on S generated by a collection of totally bounded continuous pseudometrics has a continuous extension to X.*

(7) *S is precompact uniformly embedded in X.*

Proof. We have not been able to give a proof of this theorem in the same fashion as that of Theorem 4.8. We will thus proceed to show that (4) implies (5) implies (2) implies (3) implies (4) implies (1) implies (5); (1) implies (6) implies (7) implies (1).

(4) implies (5). This implication is immediate.

(5) implies (2). Assume (5) and suppose $\mathscr{U}$ is a finite (normal) cozero-set cover of S. By 5.2 there exists a normal sequence $(\mathscr{V}_i)_{i\in\mathbf{N}}$ of open covers of S such that $\mathscr{V}_1$ refines $\mathscr{U}$ and such that, for each $i \in \mathbf{N}$, $\mathscr{V}_i$ is finite. Then, by 4.3 there exists a continuous pseudometric d on S that is associated with $(\mathscr{V}_i)_{i\in\mathbf{N}}$ and, by 5.1, d is totally bounded. Therefore, by (5), there is a continuous pseudometric $\bar{d}$ on X such that $\bar{d} \mid S \times S = d$. Let $\mathscr{W}' = (S_{\bar{d}}(x, 2^{-4}))_{x\in X}$. Since $(X, \bar{d})$ is a pseudometric space, it is paracompact, so there is a locally finite open cover $\mathscr{W}$ of X such that $\mathscr{W}$ refines $\mathscr{W}'$. By 4.1, 4.2, and the fact that a locally finite cozero-set cover is normal, it follows that $\mathscr{W}$ is a normal open cover of X relative to the given topology on X and one easily verifies that $\mathscr{W} \mid S$ refines $\mathscr{U}$.

(2) implies (3). This implication follows from ([18], Theorem 1.2) and 4.4. In particular a cover is normal if and only if it has a locally finite cozero-set refinement.

(3) implies (4). This proof can be patterned after the proof of (4) implies (5) in Theorem 4.8 with proper modifications for the requirement of totally bounded (see [1], Theorem 2.7).

To show (4) implies (1), we state the following two lemmas.

5.4. Lemma. ([11], 15E.1). *Suppose that X is a topological space and that d is a continuous pseudometric on X. Then d is totally bounded if and only if for each $\varepsilon > 0$, X is a finite union of zero-sets of diameter at most ε.*

5.5. Lemma. *Suppose that S is T-embedded in X. If $f \in C^*(S)$, if $Z_S(f) \neq \emptyset$, and if $f \geqq 0$, then there exists $g \in C^*(X)$ such that $g \mid S = f$.*

Proof. Let $f \in C^*(S)$ and suppose $f \geqq 0$ and that $Z = Z_S(f) \neq \emptyset$. Define $\Psi_f : S \times S \to \mathbf{R}$ by

$$\Psi_f(x, y) = |f(x) - f(y)| \quad (x, y \in S).$$

Then Ψ_f is a continuous pseudometric on S. To show that Ψ_f is totally bounded, let $\varepsilon > 0$ and choose $k \in \mathbf{N}$ such that $f(x) \leqq (k + 1)\,\varepsilon$ for all $x \in S$. For $n = 1, \ldots, k$, let

$$Z_n = \{x \in S : n\varepsilon \leqq f(x) \leqq (n + 1)\,\varepsilon\}.$$

Then $(Z_1, \ldots, Z_k)$ is a finite number of zero-sets of diameter at most ε and one easily verifies that $S = \bigcup_{n=1}^{k} Z_n$. Since S is T-embedded in X, there exists a continuous totally bounded pseudometric d on X such that $d \mid S \times S = \Psi_f$. Let $g : X \to \mathbf{R}_1$ be defined by $g(x) = \inf_{y \in Z} d(x, y)$ $(x \in X)$. Then $g \in \mathscr{C}^*(X)$ and $g \mid S = f$.

(4) implies (1). Assume $S \neq \emptyset$ and let $f \in C^*(S)$. Fix an arbitrary $a \in S$, let $f(a) = \alpha$, let $g = (f \vee \alpha) - \alpha$ and $h = -((f \wedge \alpha) - \alpha)$. By 5.5, there exist $\bar{g}, \bar{h} \in C(X)$ such that $\bar{g} \mid S = g$ and $\bar{h} \mid S = h$. Let $k = (\bar{g} - \bar{h}) + \alpha$. Then one easily verifies that $k \in C(X)$ and that $k \mid S = f$.

To show (1) implies (4) we will first show the following lemma.

5.6. Lemma. *If S is a dense C^*-embedded subset of a topological space X, then S is T-embedded.*

Proof. Let $(G_1, \ldots, G_n)$ be a finite normal open cover of S. Then there is a cozero-set cover $(U_1, \ldots, U_n)$ of S such that $\mathrm{cl}_S\, U_i$ is completely separated (in S) from $S - G_i$ for $i = 1, \ldots, n$ ([18], Theorem 1.2). Since S is C^*-embedded in X, $\mathrm{cl}_S\, U_i$ is completely separated (in X) from $S - G_i$. Hence there is an $f_i \in C^*(X)$ such that $f_i(\mathrm{cl}_S\, U_i) = \{0\}$ and $f_i(S - G_i) = \{1\}$. For each $i = 1, \ldots, n$, let $Z_i = Z(f_i)$ and let $V_i = \{x \in X : f_i(x) < \frac{1}{2}\}$. We note that $V_i \cap S \subset G_i$ and $\mathrm{cl}_X\, U_i \subset \mathrm{cl}_X\, Z_i = Z_i \subset V_i$. Therefore $X = \mathrm{cl}_X\, S = \mathrm{cl}_X\big(\bigcup_{i=1}^{n} U_i\big) = \bigcup_{i=1}^{n} \mathrm{cl}_X\, U_i \subset \bigcup_{i=1}^{n} V_i$ and hence $(V_1, \ldots, V_n)$ is a cozero-set cover of X that, on S, refines $(G_1, \ldots, G_n)$. Since (4) is equivalent to (3) it follows that S is T-embedded in X.

(1) implies (5). Assume S is C^*-embedded in X. Then $\mathrm{cl}_{\beta X}\, S = \beta S$. Moreover S is C^*-embedded in βS so, by Lemma 5.6, S is T-embedded in βS. But βS is a closed subset of the normal space βX, so βS is C-embedded in βX. Now any totally bounded continuous pseudometric d on βS is $\aleph_0$-separable. Hence by Theorem 4.8.5 d has a continuous pseudometric extension to βX.

(1) implies (6). Let $\mathscr{U}$ be an admissible uniformity on S generated by a collection of totally bounded continuous pseudometrics. By the above equivalences if S is C^*-embedded in X then every totally bounded continuous pseudometric on S has a continuous extension to X. Proceeding as in the proof of the corresponding implication in Theorem 4.8, we have that statement (1) implies statement (6) of the present Theorem.

If $\mathscr{U}$ is an admissible uniformity on S generated by a collection $\mathscr{F}$ of bounded continuous real valued functions on S then the associated pseudometrics ψ_f for $f \in \mathscr{F}$ are totally bounded. Thus $\mathscr{U}$ is a uniformity generated by a collection of totally bounded continuous pseudometrics. Hence statement (6) implies statement (7).

It remains to show that statement (7) implies statement (1). Here the proof proceeds the same as in the proof of the corresponding implication of Theorem 4.8 with the following modifications. Of course $\mathscr{C}(S)$ is replaced by $\mathscr{C}^*(S)$ and "continuous real valued function" by "bounded continuous real valued function". The stronger result of Katětov (also referred to in the proof of Theorem 2.1) is needed. In particular every bounded uniformly continuous real valued function on $(S, \mathscr{C}^*(S))$ has a bounded uniformly continuous real valued extension to $(X, \mathscr{U})$. This completes the proof.

5.7. Corollary. *The Čech-Stone compactification βX of X is that unique compact Hausdorff space containing X densely such that every admissible precompact uniformity on X has a continuous extension.*

6. Normal Spaces

Using our characterizations stated previously we can now give some new characterizations of normal spaces.

We give the new results along with some already known for the sake of completeness.

6.1. Theorem. *For completely regular spaces $(X, \mathscr{T})$ the following statements are equivalent:*

(1) *$(X, \mathscr{T})$ is normal.*

(2) *Every closed subset is precompact uniformly embedded in X.*

(3) *Every closed subset is prerealcompact uniformly embedded in X.*

(4) *For every closed subset F of X, every $\aleph_0$-separable continuous pseudometric on F has a continuous pseudometric extension to X.*

(5) *For every closed subset F of X, every totally bounded continuous pseudometric on F has a continuous pseudometric extension to X.*

(6) *For every closed subset F of X, every finite normal cozero-set cover of F has a refinement that extends to a normal open cover of X.*

(7) *For every closed subset F of X every countable star-finite cozero-set cover of F has a refinement that can be extended to a countable cozero-set cover of X.*

(8) *Every countable locally finite cozero-set cover of each closed subset has a refinement that can be extended to a normal open cover of X.*

(9) *Every countable normal open cover of each closed subset has a refinement that can be extended to a countable locally finite cozero-set cover of X.*

Proof. This is just a restatement of the Tietze Extension Theorem in conjunction with our results in this paper.

7. Closing Remarks

The relationship between C-embedding and C^*-embedding is obvious. If S is a C-embedded subset of a topological space X then S is a C^*-embedded subset of X. On the other hand, if S is uniformly embedded (or P-embedded) then S is C-embedded in X.

7.1. Lemma. *Let f be a continuous real valued function on the P-embedded subset S of $(X, \mathscr{T})$. If the zero-set Z on S of f is non-empty and if $f(x) \geqq 0$ for all x then there is a continuous real valued function g on X such that $g \mid S = f$.*

Proof. For f as in the hypothesis it is easy to see that the function

$$\psi_f : S \times S \to \mathbf{R}^+ \quad \text{defined by} \quad \psi_f(x, y) = |f(x) - f(y)|, \quad (x, y \in S),$$

is a continuous pseudometric on $S \times S$. Since S is P-embedded in X, there is a continuous pseudometric d on X such that $d \mid S \times S = \psi_f$. Let $g : X \to \mathbf{R}^+$ be defined by $g(x) = d\text{-dist}\,(x, Z)$, $(x \in X)$. Clearly g is a continuous real valued function on X and, if $x \in S$,

$$g(x) = d\text{-dist}\,(x, Z) = \psi_f\text{-dist}\,(x, Z) = \inf_{a \in Z} |f(x) - f(a)| = f(x).$$

The proof is complete.

7.2. Theorem. *If S is uniformly embedded in X, then S is C-embedded in X.*

Proof. Assume $S \neq \emptyset$ and f is a continuous real valued function on S. Let a be in S and $f(a) = \alpha$. Then $f \vee \alpha, f \wedge \alpha$, $g = (f \vee \alpha) - \alpha$ and $h = -((f \wedge \alpha) - \alpha)$ are continuous real valued functions on S with $g \geqq 0$, $h \geqq 0$, the zero set on S of g, $Z_S(g) \neq \emptyset$, and $Z_S(h) \neq \emptyset$. By 7.1 there are continuous real valued extensions $\bar{g}$ and $\bar{h}$ to X of g and h respectively. Then $\bar{f} = (\bar{g} - \bar{h}) + \alpha$ is the required extension of f.

In ([6], Example G), Bing has given an example of a normal topological space that is not collectionwise normal. We briefly state this example here. Let Y be any set such that card $Y = \aleph_1$ and let $\mathscr{P}(Y)$ be the power set of Y. For each $B \in \mathscr{P}(Y)$, let $X_B = \{0, 1\}$ and let $X = \prod_{B \in \mathscr{P}(Y)} X_B$. For each $y \in Y$ let $f_y \in X$ defined by $f_y(B) = 1$ if for $B \in \mathscr{P}(Y)$, $y \in B$ and $f_y(B) = 0$ if for $B \in \mathscr{P}(Y)$, $y \notin B$. Let $X_0 = \{f_y : y \in Y\}$. For each $y \in Y$ and finite subcollection $\mathscr{F}$ of $\mathscr{P}(Y)$, set

$$G(f_y, \mathscr{F}) = \{f \in X : f(B) = f_y(B) \text{ for each } B \in \mathscr{F}\} ,$$

and set

$$\mathscr{G} = \{G(f_y, \mathscr{F}) : y \in Y, \mathscr{F} \subset \mathscr{P}(Y), \mathscr{F} \text{ finite}\} .$$

A topology $\mathscr{T}$ on X is defined to be the unique topology on X having for a base the collection

$$\mathscr{G} \cup \{\{f\} : f \in X \setminus X_0\} .$$

Bing shows that X equipped with this topology is a normal T_1 space and that X_0 is a closed discrete subset of X. He also shows that there exists no pairwise disjoint family $(G_y)_{y \in Y}$ of open subsets of X such that $f_y \in G_y$ for each $y \in Y$. Hence X is not collectionwise normal. Since X is normal X_0 is C-embedded in X. However X_0 is not uniformly embedded (P-embedded) in X by Corollary 3.2.

In [11] Gillman and Jerison give the example of a C^*-embedded subset of a completely regular space X that is not C-embedded. They take $S = \mathbf{N}$, the natural numbers and X to be $\beta\mathbf{R} \setminus (\beta\mathbf{N} \setminus \mathbf{N})$.

In [19], it is shown when the notions of P and C-embedding are equivalent.

In closing we remark that for infinite cardinal numbers γ one can define the concepts of P^γ-embedding (see [19]) and γ-uniformly-embedding using the uniformities $\mathscr{U}_\gamma$ (see [4] and [10]). Here a γ-uniformly embedded subset would be one for which every admissible uniformity on S generated by a collection of γ-separable continuous pseudometrics on S has a continuous extension. With an appropriate modification of the lemma, it is possible to show that *S is γ-uniformly embedded in X if and only if S is P^γ-embedded in X.*

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