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ON SOME KLEIN-GORDON-SCHRÖDINGER TYPE SYSTEMS

NIKOLAOS M. STAVRAKAKIS*

Abstract. We present some recent trends in the theory of Klein-Gordon-Schrödinger type Systems. Then we give some resent results on the following special type of a dissipative Klein-Gordon-Schrödinger System

$$\begin{aligned} &\mathrm{i}\,\psi_t + \kappa\psi_{xx} + \mathrm{i}\,\alpha\psi = \phi\psi, && x \in \Omega, \quad t > 0,\\ &\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t = -\operatorname{Re}\psi_x, && x \in \Omega, \quad t > 0, \end{aligned}$$

satisfying the initial and boundary conditions

$$\begin{aligned} &\psi(x,0) = \psi_0(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), && x \in \Omega,\\ &\psi(x,t) = \phi(x,t) = 0, && x \in \partial\Omega, \quad t > 0, \end{aligned}$$

with $\kappa,\ \alpha,\ \lambda$ positive constants and Ω a bounded subset of $\mathbb{R}$. This certain system describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field. Global existence and uniqueness of solutions are derived. Also necessary conditions for the exponential energy decay of the system are established. Finally, we mention some resent results concerning the asymptotic behavior of this problem under external forces in dimension 1.

Key words. Klein-Gordon-Schrödinger system, Electron-Ion Plasma Waves, Dissipation, Global Existence, Uniqueness, Energy Decay

1. Prehistory and Modelling. Systems of Klein-Gordon-Schrödinger type have been studied for many years. To our knowledge, it seems that the first problems of this type is the so called *Yukawa System* (see, Yukawa H. [36]), which goes back to 1935 and is of the following form

$$\mathrm{i}\,\psi_t + \frac{1}{2}\Delta\psi = -\phi\psi, \qquad x \in \Omega, \quad t > 0, \quad \Omega \subseteq \mathbb{R}^N, \tag{1.1}$$

$$\phi_{tt} - \Delta\phi = |\psi|^2, \qquad x \in \Omega, \quad t > 0, \tag{1.2}$$

with initial conditions

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \tag{1.3}$$

where ψ is the complex nucleon field, ϕ is the real meson field. The system (1.1)–(1.3) has been examined with respect to the existence of local and global solutions, blow-up, exponential decay, and global attractor in both bounded and unbounded domains of $\mathbb{R}^N$, for $N \leq 3$. I. Fukuda and M. Tsutsumi in a series of papers (see [11], [12], [13], [14]) have studied existence of local and global solutions as well as uniqueness, decay estimates and blow-up of solutions in one and several dimensions. A. Bachelot [1], study existence of local and global solutions and uniqueness. N. Hayashi and W. von Wahl, [23] study the existence of global strong solutions of a coupled Klein-Gordon-Schrödinger system. Biler P. [2], establishes the existence of attractors in a weak topology for a system of Schrödinger and Klein-Gordon equations with Yukawa coupling. M. Ohta [29] obtained stability results. B. Guo and Y. Li, [21], proved the existence of a strong global attractor in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ attracting bounded sets of $H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. Cavalcanti

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M. M. and Cavalcanti V. N. D. [6], prove the global existence and uniform exponential decay of solutions for a coupled Klein-Gordon-Schrödinger system of a generalized Yukawa type. K. Lu and Wang B. [25], established the existence of a strong global attractor in $H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$, $N = 1, 2, 3$, attracting bounded sets of $H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$, $k \geq 1$. Quite recently, H. Pecher [31] proved the existence of a global solution of the Klein-Gordon-Schrödinger system with Yukawa coupling and rough data, which have not finite energy necessarily.

An other model which is of the same type is the so called *Zakharov System*, which is formed by V. E. Zakharov [39] in early seventies and is of the form

$$\mathrm{i}\,\psi_t + \kappa\Delta\psi = \phi\psi, \qquad x \in \Omega, \quad t > 0, \quad \Omega \subseteq \mathbb{R}^N \tag{1.4}$$

$$\phi_{tt} - \Delta\phi = \Delta|\psi|^2, \qquad x \in \Omega, \quad t > 0, \tag{1.5}$$

with initial conditions

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \tag{1.6}$$

where ψ is the (complex) envelop of the electric field, ϕ is the (real) fluctuation of the ion density about the equilibrium value. The system (1.4)–(1.6) has been examined with respect to the existence of local and global solutions, blow-up, exponential decay and global attractor in both bounded and unbounded domains of $\mathbb{R}^N$, for $N \leq 3$. Flahaut I. [10], proves the existence of a weak global attractor in $H_0^1((0,L)) \times H_0^1((0,L)) \bigcap H^2((0,L)) \times H_0^1((0,L)) \bigcap H^3((0,L))$ and gives an estimation of the Hausdorff and Fractal dimension for the dissipative Zakharov system. L. Glangetas and F. Merle, [16], [17] study the existence of self-similar blow-up solutions as well as concentration properties of blow-up solutions and instability results for a Zakharov system in $\mathbb{R}^2$. Guo Boling and Yongsheng Li [21] proves the existence of attractor for a dissipative Klein-Gordon-Schrödinger system of Zakharov type in $\mathbb{R}^3$. J. Bourgen [3], studies the Cauchy and invariant measure problem for the periodic Zakharov system. Also in an AMS monograph [4] J. Bourgen presents in a more general setting resent results on global solutions of nonlinear Schrödinger equations. J. Bourgen and J. Colliander, [5] study the wellposedness of the Zakharov System. J. Colliander [8], in his thesis, presents an extensive study of the initial value problem for the Zakharov system. J. Ginibre, Y. Tsutsumi and G. Velo [15] obtain certain results on the Cauchy problem for the Zakharov System. O. Goubet and Moise I., [18], establishe the existence of attractor for dissipative Zakharov system in bounded domains. Takaoka H. [35] proves the well-posedness for the Zakharov system with periodic boundary conditions. Masselin Vincent A, [27] establish results on the blow-up rate for the Zakharov system in Dimension 3. J. Colliander and G. Staffilani [9], obtain regularity bounds on Zakharov systems. H. Pecher [30] proves global well-posedness below energy space for the 1-dimensional Zakharov system. A. Grünrock and Pecher H., [20], establish bounds in time for the Klein-Gordon-Schrödinger and the Zakharov systems.

Here we consider the following system of *Klein-Gordon-Schrödinger Type*

$$\mathrm{i}\,\psi_t + \kappa\psi_{xx} + \mathrm{i}\,\alpha\psi = \phi\psi, \qquad x \in \Omega, \quad t > 0, \tag{1.7}$$

$$\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t = -\operatorname{Re}\psi_x, \qquad x \in \Omega, \quad t > 0, \tag{1.8}$$

with initial and boundary conditions

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \tag{1.9}$$

$$\psi(x,t) = \phi(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{1.10}$$

where $\kappa > 0$, $\alpha > 0$, $\lambda > 0$ and Ω is a bounded subset of $\mathbb{R}$, i.e. a bounded interval. Here ψ stands for the dimensionless *low frequency electric field* and ϕ for (the dimensionless) *(real) low frequency density*. Problem (1.7)–(1.10) models the *Upper Hybrid Heating* (UHH) scheme for plasmas in fusion devices. (UHH) is the dominant branch of the general *Electron Cyclotron Resonance Heating* (ECRH) scheme, which, for Tokamaks and Stellarators, constitutes a basic method of plasma build-up and heating. The celebrated Zakharov system, is highly successful in a multitude of applications, such as laser fusion, electron beam fusion, solar radio bursts etc, e.g., see Sulem, C., Sulem, P. L. [34]. However, regarding the study of (UHH), the Zakharov system may not be implemented for the following reasons: (i) It does not consider the effect of collisions. Therefore, it can only describe the collisionless part of the damping. (ii) It is indifferent to the presence of a dc magnetic field, due to the nature of the ponderomotive force. Therefore, perpendicular waves, peculiar to (UHH) heating, cannot be modelled.

In order to overcome these shortcomings, *we study the effect of the space-time varying electric field on the ion channel*. Specifically, we consider the drift motion of the ions caused by the time variation of the electric field, namely the *polarization drift*. However, *the space variation of the electric field is included in the polarization drift*. Indeed, it turns out that the contribution of this effect to the system of equations involves the space derivative of the electric field. In this respect, we may talk about a *non-homogeneous* polarization drift (note that a homogeneous time-varying field is sufficient for the standard polarization drift to occur). This drift induces a *polarization current*, which plays the role of the low frequency coupling between ions and electrons (see, J. Wesson [38] and D. R. Nicholson [28]).

2. Mathematical Analysis.

(A) Estimates – Global Existence – Uniqueness

All the results presented in the rest of the paper are included in the work [24]. Introduce the new real variable $\theta = \phi' + \delta\phi$, where $\delta > 0$ to be specified later. System (1.7)–(1.10) becomes

$$\mathrm{i}\,\psi' + \kappa\psi_{xx} + \mathrm{i}\,\alpha\psi = \phi\psi, \tag{2.1}$$

$$\phi' + \delta\phi = \theta, \tag{2.2}$$

$$\theta' + (\lambda - \delta)\theta - \phi_{xx} + (1 - \delta(\lambda - \delta))\phi = -\operatorname{Re}\psi_x, \tag{2.3}$$

where $x \in \Omega$, $t > 0$, and initial boundary conditions:

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \theta(x,0) = \theta_0(x), \tag{2.4}$$

$$\psi(x,t) = 0 \qquad x \in \partial\Omega, \quad t > 0. \tag{2.5}$$

Then the main Existence and Uniqueness Theorem states as follows

Theorem 2.1 (Global Existence & Uniqueness Theorem). *Assume that the initial conditions* $(\psi_0, \phi_0, \theta_0) \in \left(H_0^1 \cap H^2(\Omega)\right)^2 \times H_0^1(\Omega)$. *Then, there exists a unique solution* (ψ, ϕ, θ) *for the problem* (2.1)–(2.5) *such that*

$$\psi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)), \quad \psi_t \in L^\infty(0,\infty; L^2(\Omega)),$$
$$\phi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)), \quad \phi_t \in L^\infty(0,\infty; H_0^1(\Omega)), \quad \phi_{tt} \in L^\infty(0,\infty; L^2(\Omega)),$$

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \qquad x \in \Omega. \tag{2.6}$$

Proof. The proof is based on two *a priori* estimates. The first *a priori* estimate is derived for $(\psi, \phi, \theta) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ and the second is for $(\psi, \phi, \theta) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$. Then the existence and uniqueness results follow applying certain functional analytic methods. □

(B) Energy Decay

We return to the original system (1.7)–(1.10) and define the corresponding *energy functional* as

$$E(t) = \frac{1}{2}\left(||\psi||^2 + \kappa||\psi_x||^2 + \int \phi|\psi|^2\,\mathrm{d}x + ||\phi'||^2 + ||\phi_x||^2 + ||\phi||^2\right).$$

Note that, as in the Zakharov setup, the integral $\int \phi|\psi|^2\,\mathrm{d}x$ determines the sign of the Hamiltonian. Nevertheless, this integral cannot possibly affect the asymptotic value of the energy, which remains positive, as indicated by the following result:

Lemma 2.2. *Let* Theorem 2.1 *be fulfilled. Let that there exists $R > 0$, such that $||E(0)|| \leq R$. Then, there exists $t^* > 0$ such that $E(t) > 0$, for all $t \geq t^*$.* We are ready to state the final result of this part.

Theorem 2.3. *Suppose that for the parameters κ, λ, α condition $2\lambda\alpha\kappa > 1$ holds and there exists $R(\kappa, \lambda, \alpha) > 0$, satisfying condition $R^2 < 2\lambda\kappa\alpha - 1$, such that $||E(0)|| \leq R$. Then, the problem* (1.7)–(1.10) *manifests energy decay.*

The time t^* introduced in the energy decay analysis has a specific physical meaning. This is the time so that the non-collisional integral $\int \phi|\psi|^2\,\mathrm{d}x$ is absorbed by the collisional terms. Given standard reaction conditions, this time is of the order of 10^{-8}–10^{-6} seconds.

3. Discussion and Open Problems. The physical interpretation of the results ends up with a *threshold of the effectiveness of UHH*, involving the plasma variables, i.e., density, ion and electron temperatures as well as the magnetic field. Our result suggests that *UHH is favored by high-density conditions*, such as in the very promising density-limit shots, where also the *temperature assumes relatively low values.*

Possible generalizations, include the study of a similar model in a *bounded or an unbounded domain* and the generalization of the results in *more regular spaces.* Also *Stability results* for this KGS-type evolution system, e.g., existence and dimension of attractor are under consideration. Finally, more complicate forms, in bounded and unbounded domains, belong to future plans. Recently, we got some further results concerning the asymptotic behavior of solutions of the problem (1.7)–(1.10). Actually, we have proved the existence of a *strong global attractor* for the problem (1.7)–(1.10) under certain external forces, when it is defined in a bounded domain $D \subset \mathbb{R}$ (see, work [32]). Also, some upper estimates are fixed for the Hausdorff and Fractal dimensions of this strong global attractor (see, work [33]).

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