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EIGENVALUE QUESTIONS ON SOME QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. We present resent results on some quasilinear elliptic problems of p-Laplacian type. Among other things we prove the existence of a positive principal eigenvalue for a p-Laplacian equation and discuss questions of simplicity and isolation of the eigenvalue.

Key words. Quasilinear Elliptic Problems, Unbounded Domains, Indefinite Weights, Spectral Theory, Isolation, Principal Eigenvalues.

AMS subject classifications. 35B20, 35B32, 35B40, 35B45, 35B65, 35D05, 35D10, 35J50, 35J70, 35P30.

1. Introduction. In this paper we prove the existence of a positive principal eigenvalue of the following quasilinear elliptic problem,

$$-\Delta_p u(x) = \lambda g(x)|u|^{p-2}u, \qquad x \in \mathbb{R}^N, \tag{1.1}$$

$$\lim_{|x|\to+\infty} u(x) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$. Next, we state the general hypotheses which will be assumed throughout the paper:

($\mathcal{E}$) *Assume that* N, p *satisfy the following relation* $N > p > 1$.

($\mathcal{G}$) g *is a smooth function, at least* $C^{1,\alpha}(\mathbb{R}^N)$ *for some* $\alpha \in (0,1)$, *such that* $g \in L^\infty(\mathbb{R}^N)$ *and* $g(x) > 0$, *on* Ω^+, *with measure of* Ω^+, $|\Omega^+| > 0$. *Also there exist* R_0 *sufficiently large and* $k > 0$ *such that* $g(x) < -k$, *for all* $|x| > R_0$.

On various types of bounded domains the picture for "the principal eigenpair" seems to be fairly complete where for unbounded domain, papers have appeared quite recently. These problems are of a more complex nature, as the equation may give rise to a noncompact operator (see [4]).

The main aim of this paper is to study the quasilinear elliptic problem (1.1)–(1.2), by generalizing ideas introduced in the paper [7]. In Section 2, we study the space setting of the problem (1.1)–(1.2). A generalised version of Poincaré's inequality plays a crucial role. In Section 3, we define the basic operators for the construction of the first positive eigenvalue the proof which is based on Ljusternik-Schnirelmann's theory. Also here, we derive some regularity results. Finally, in Section 4, we establish the simplicity and isolation of the principal eigenvalue. The detailed proofs of the results appearing here are presented in the work [5].

Notation. We *denote* by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R and $B_R^* =: \mathbb{R}^N \setminus B_R$. For *simplicity reasons* sometimes we use the symbols C_0^∞, L^p, $W^{1,p}$

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respectively for the spaces $C_0^\infty(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, $W^{1,p}(\mathbb{R}^N)$ and $||.||_p$ for the norm $||.||_{L^p(\mathbb{R}^N)}$. Also, sometimes when the domain of integration is not stated, it is assumed to be all of $\mathbb{R}^N$. Equalities introducing definitions are denoted by "=:". Denote by $g_\pm =: \max\{\pm g, 0\}$. The end of the proofs is marked by "$\lhd$".

2. Space Setting. In this section we are going to characterize the space $\mathcal{V}_g$ (introduced below) in terms of classical Sobolev spaces. Let B be a ball centered at the origin of $\mathbb{R}^N$, such that $\int_B g(x)\,\mathrm{d}x < 0$ and $g(x) \le -k$, for all $x \in B^*$. First, we prove the following type of Poicaré's inequality:

Theorem 2.1. *Suppose* $\int_{\mathbb{R}^N} g(x)\,\mathrm{d}x < 0$. *Then there exists* $\alpha > 0$, *such that*

$$\int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x > \alpha \int_{\mathbb{R}^N} g(x)|u|^p\,\mathrm{d}x,$$

for all $u \in W^{1,p}(\mathbb{R}^N)$, *such that* $\int_{\mathbb{R}^N} g(x)|u|^p\,\mathrm{d}x > 0$.

By the above result we may introduce the following norm

$$\|u\|_g =: \left(\int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x - \frac{\alpha}{2}\int_{\mathbb{R}^N} g(x)|u|^p\,\mathrm{d}x\right)^{1/p}. \tag{2.1}$$

We define the space $\mathcal{V}_g$ to be the completion of C_0^∞ with respect to the norm $\|.\|_g$. Let $\mathcal{V}_g^*$ be the dual space of $\mathcal{V}_g$ with the pairing $(.,.)_\mathcal{V}$. Note that $\mathcal{V}_g$ is a uniformly convex Banach space. Although the space $\mathcal{V}_g$ would seem to depend on g, we shall prove that the space is independent of g. To achieve this result we need the following three results.

Corollary 2.2. *Under the assumptions of* Theorem 2.1, *for all* $u \in C_0^\infty(\mathbb{R}^N)$, *we have:*

$$(i) \qquad \int_{\mathbb{R}^N} |\nabla u|^p \le 2\|u\|_g^p, \tag{2.2}$$

$$(ii) \qquad \left|\int_{\mathbb{R}^N} g|u|^p\,\mathrm{d}x\right| \le \frac{2}{\alpha}\|u\|_g^p. \tag{2.3}$$

Lemma 2.3. *Assume that the hypotheses of* Theorem 2.1 *are valid. Let* $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ *be a bounded sequence in* V_g. *Then* $\{\int_B g|u_n|^p\,\mathrm{d}x\}$ *is bounded in* V_g.

To prove the next results we need to introduce the following notation: $D_1 =: \{x \in B : g(x) > 0\}$, $D_2 =: \{x \in B : g(x) \le 0\}$ and

$$\bar{g}(x) =: \begin{cases} g_+(x), & x \in D_1, \\ -g_-(x), & x \in D_2. \end{cases}$$

Lemma 2.4. *Assume that the hypotheses of* Theorem 2.1 *are valid. Then there exist constants* $K_0 > 0$ *and* $K_1 > 0$ *such that*

$$(i) \qquad \int g_+(x)|u|^p\,\mathrm{d}x \le K_0\|u\|_g^p, \tag{2.4}$$

$$(ii) \qquad -\int g_-(x)|u|^p\,\mathrm{d}x \le K_1\|u\|_g^p, \tag{2.5}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.
Next, we give the following uniform Sobolev characterization of the space V_g.

PROPOSITION 2.5. *Suppose that* g *satisfies* $(\mathcal{G})$. *Then* $\mathcal{V}_g = W^{1,p}(\mathbb{R}^N)$.

3. Principal Eigenvalue and Regularity Results. In this section we are going to define the basic operators and some of their characteristics, which will help to prove the existence of a positive principal eigenvalue of the problem (1.1)–(1.2). Finally, we close this section by proving some regularity results.

For any r_0 large enough ($r_0 \geq R_0$), there exists $\sigma_0 > 0$, such that $g(x) \leq -\frac{k}{\sigma_0}$, for all $|x| \geq r_0$. For later needs we introduce the following smooth splitting of the weight function g

$$g_2(x) =: \begin{cases} g(x), & \text{for } |x| \geq r_0, \\ -\dfrac{k}{\sigma_0}, & \text{for } |x| < r_0, \end{cases} \qquad \text{and} \quad g_1(x) =: g(x) - g_2(x).$$

Let us define the operator $A_\lambda : D(A_\lambda) \subset W^{1,p} \to W^{1,q}$ as follows

$$(A_\lambda(u), v) = \int (|\bigtriangledown u|^{p-2} \bigtriangledown u \bigtriangledown v - \lambda g_2 |u|^{p-2} uv)\, \mathrm{d}x, \qquad \text{for all } u, v \in W^{1,p}.$$

We can then define the bilinear mapping

$$a_\lambda : W^{1,p} \times W^{1,p} \to \mathbb{R}, \text{ by } a_\lambda(u,v) =: (A_\lambda(u), v).$$

It is easy to see that a_λ is bounded and coercive for all $u, v \in D(A_\lambda)$ and $\lambda > \lambda_0$.

Next, we introduce the following bilinear form $b(u,v)$

$$b(u,v) = \int g_1 |u|^{p-2} uv\, \mathrm{d}x, \qquad \text{for all } u, v \in W^{1,p}(\mathbb{R}^N).$$

We see that $b(u,v)$ is bounded by using Hölder's inequality and the definition of g_1, for all $u, v \in W^{1,p}$.

Therefore by the Riesz Representation Theory we can define a linear operator $B_\lambda : D(B_\lambda) \subset L^p \longmapsto L^q$, such that $(B_\lambda(u), v) = b(u,v)$, for all $u, v \in D(B_\lambda)$ and $\lambda > 0$. It is easy to see that $D(B_\lambda) \subset W^{1,p}$. Moreover it is easy to see that the operators A_λ, B_λ are well defined and A_λ is continuous.

LEMMA 3.1.
(i) *if* $\{u_n\}$ *is a sequence in* $W^{1,p}$, *with* $u_n \rightharpoonup u$, *then there is a subsequence, denoted again by* $\{u_n\}$, *such that* $B_\lambda(u_n) \to B_\lambda(u)$,
(ii) *if* $B_\lambda'(u) = 0$, *then* $B_\lambda(u) = 0$.

THEOREM 3.2. *Let* $1 < p < N$. *Assume that* g *satisfies* $(\mathcal{G})$. *Then*
(i) *the problem* (1.1)–(1.2) *has a sequence of solutions* (λ_k, u_k) *with* $\int g(x)|u_k|^p = 1$, $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \to \infty$, *as* $k \to \infty$,
(ii) *the eigenfunction* u_1 *corresponding to the first eigenvalue can be taken positive in* $\mathbb{R}^N$.

Proof. The proof is based on Ljusternik-Schnirelmann theory. □

The next theorem examines the regularity as well as the L^{p_k} character and asymptotic behavior of the $W^{1,p}$ solutions of the problem (1.1)–(1.2).

THEOREM 3.3. *Suppose that* $u \in W^{1,p}$ *is a solution of the problem* (1.1)–(1.2). *Then* $u \in L^{p_k}$, *for all* $p_k \in [pc, +\infty]$ *and the solutions* $u(x)$ *decay uniformly to zero, as* $|x| \to +\infty$.

COROLLARY 3.4. *For any* $r > 0$, *the solutions of the problem* (1.1)–*1.2 belong to* $C^{1,\alpha}(B_r)$, *where* $\alpha = \alpha(r) \in (0,1)$.

4. Simplicity and Isolation of the Principal Eigenvalue. In this section, first we are going to prove the simplicity of the principal eigenvalue of the problem (1.1)–(1.2) by generalizing Picone's identity.

THEOREM 4.1 (Generalized Picone's Identity). *Let* $v > 0$, $u \geq 0$ *be differentiable functions in* Ω, *where* Ω *is a bounded or unbounded domain in* $\mathbb{R}^N$. *Denote by*

$$L(u,v) = |\bigtriangledown u|^p + (p-1)\frac{u^p}{v^p}|\bigtriangledown v|^p - p\frac{u^{p-1}}{v^{p-1}} \bigtriangledown u|\bigtriangledown v|^{p-2} \bigtriangledown v,$$

$$R(u,v) = |\bigtriangledown u|^p - \bigtriangledown\left(\frac{u^p}{v^{p-1}}\right)|\bigtriangledown v|^{p-2} \bigtriangledown v.$$

Then $L(u,v) = R(u,v) \geq 0$. *Moreover,* $L(u,v) = 0$, *a.e. in* Ω, *if and only if* $\bigtriangledown(u/v) = 0$, *a.e. in* Ω, *i.e.,* $u = kv$, *for some constant* k *in each component of* Ω.

Proof. For the proof we refer to W. Alegretto and Y. X. Huang [1, Theorem 1.1]. □

THEOREM 4.2. *Suppose* $v \in C^1$ *satisfies* $-\Delta_p v \geq \lambda g v^{p-1}$ *and* $v > 0$ *in* $\mathbb{R}^N$, *for some* $\lambda > 0$. *Then, for* $u \geq 0$ *in* $W^{1,p}$ *we have*

$$\int |\bigtriangledown u|^p \,\mathrm{d}x \geq \lambda \int g(x)|u|^p \,\mathrm{d}x, \tag{4.1}$$

and $\lambda \leq \lambda_1^+$. *The equality in* (4.1) *holds if and only if* $\lambda = \lambda_1^+$, $u = kv$ *and* $v = cu_1$, *for some constants* k, c. *In particular, the principal eigenvalue* λ_1^+ *is simple.*

THEOREM 4.3. *The principal eigenvalue* λ_1 *of the problem* (1.1)–(1.2) *is isolated in the following sense: there exists* $\eta > 0$, *such that the interval* $(-\infty, \lambda_1 + \eta)$ *does not contain any other eigenvalue than* λ_1.

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