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OSCILLATION AND NONOSCILLATION CRITERIA FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Several criteria are given for having the retarded functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=\int_{-1}^{0}x(t-r(\theta))\,\mathrm{d}q(\theta)$$

either oscillatory or nonoscillatory, depending upon the smoothness of the delay function $r(\theta)$.

1. Introduction. The purpose of this note is to investigate the oscillatory behavior of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=\int_{-1}^{0}x(t-r(\theta))\,\mathrm{d}q(\theta) \tag{1.1}$$

where $x(t)\in\mathbb{R}$, $r(\theta)$ is a positive real continuous function on $[-1,0]$, and $q(\theta)$ a real function of bounded variation on $[-1,0]$, normalized in manner that $q(-1)=0$.

We will analyze the existence or nonexistence of oscillations in terms of the smoothness of the delay functions, $r(\theta)$. Namely, when $r(\theta)$ is in the set C^+ of all continuous functions in $[-1,0]$, or in D^+, the set of all differentiable functions on $[-1,0]$.

It will be also considered the relevant class of differential difference equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=\sum_{j=1}^{p}a_j x(t-r_j) \tag{1.2}$$

where the a_j are nonzero real numbers and each r_j is a positive real number $(j=1,\dots,p)$. As is well-know, this equation can be obtained from (1.1), under the assumption that $q(\theta)$ is a step function with a number p of jump points. More concretely it can be obtained from (1.1) with $q(\theta)$ given explicitly, for example, by

$$q(\theta)=\sum_{j=1}^{p}H(\theta-\theta_j)\,a_j, \tag{1.3}$$

where, for $-1<\theta_1<\ldots<\theta_p<0$, by H we mean the Heaviside function and the delays, r_j, are obtained through any function $r(\theta)\in C^+$ which satisfy $r(\theta_j)=r_j$ for $j=1,\dots,p$.

A metric is introduced in C^+ through the norm $\|r\|=\max\{r(\theta):-1\le\theta\le0\}$ $(r\in C^+)$. The value $m(r)=\min\{r(\theta):-1\le\theta\le0\}$ will also have some relevance in the sequel.

By a solution of (1.1) we mean a continuous function $x:[-\|r\|,\infty[$, which is differentiable on $[0,+\infty[$ in manner that (1.1) be satisfied for every $t\ge0$. A solution is said oscillatory whenever it has an infinite number of zeros; otherwhise it will be said

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nonoscillatory. When all solutions are oscillatory the equation (1.1) is called oscillatory. If (1.1) has at last one nonoscillatory solution the equation will be said nonoscillatory.

We will take the Banach space NBV of all (normalized) real functions of bounded variation, ϕ, on $[-1,0]$, such that $\phi(-1)=0$. Denoting by $\int_{-1}^{0}|\mathrm{d}\phi(\theta)|$ the total variation of ϕ on $[-1,0]$, through $\|\phi\|=\int_{-1}^{0}|\mathrm{d}\phi(\theta)|$, we introduce a norm in NBV.

We will say that a function $\phi:[-1,0]\to\mathbb{R}$ is increasing (decreasing) on $J\subset[-1,0]$, if ϕ is non constant on J and for every $\theta_1,\theta_2\in J$ such that $\theta_1<\theta_2$, one has $\phi(\theta_1)\le\phi(\theta_2)$ (respectively, $\phi(\theta_2)\le\phi(\theta_1)$. Following [1, 2] , a given $\theta\in[-1,0]$ is said a point of increase (respectively, a point of decrease) of ϕ, if for every $\varepsilon>0$, sufficiently small, ϕ is increasing (decreasing) in $[\theta-\varepsilon,\theta+\varepsilon]$ ($[-\varepsilon,0]$ if $\theta=0,[-1,-1+\varepsilon]$ if $\theta=-1$). If there exists a $\varepsilon>0$ such that ϕ is constant in $[\theta-\varepsilon,\theta+\varepsilon]$ ($[-\varepsilon,0]$ if $\theta=0$, $[-1,-1+\varepsilon]$ if $\theta=-1$), θ will be said a point of constancy of ϕ.

As is well known, any function $\phi\in NBV$ can be decomposed as the difference of two nondecreasing functions α and $\beta:\phi=\alpha-\beta$. This decomposition is not unique and a particular decomposition of ϕ is given by

$$\phi=\phi_+-\phi_-, \tag{1.4}$$

where by ϕ_+ and ϕ_- we denote, respectively, the positve and negative variation of ϕ, which are defined as follows. For each $\theta\in[-1,0]$, let $\mathcal{P}$ be the set off all partitions $P=\{-1=\theta_0,\theta_1,\ldots,\theta_k=\theta\}$ of the interval $[-1,\theta]$ and to each $P\in\mathcal{P}$ associate the sets

$$A(P)=\{j:\phi(\theta_j)-\phi(\theta_{j-1})>0\}\ \text{ and }\ B(P)=\{j:\phi(\theta_j)-\phi(\theta_{j-1})<0\}.$$

Then ϕ_+ and ϕ_- are given, respectively, by

$$\phi_+(\theta)=\sup\left\{\sum_{j\in A(P)}(\phi(\theta_j)-\phi(\theta_{j-1})):P\in\mathcal{P}\right\}$$

and

$$\phi_-(\theta)=\sup\left\{\sum_{j\in B(P)}|\phi(\theta_j)-\phi(\theta_{j-1})|:P\in\mathcal{P}\right\}.$$

(whenever $A(P)$ or $B(P)$ are empty, we make $\phi_+(\theta)=0,\phi_-(\theta)=0$). One easily sees that both ϕ_+ and ϕ_- are nondecreasing functions such that $\phi(\theta)=\phi_+(\theta)-\phi_-(\theta)$, for every $\theta\in[-1,0]$.

The oscillatory analysis of the equation (1.1) can be made, as is well known (see [3]), through the study of the zeros of the function $F(\lambda)=\lambda-\int_{-1}^{0}\exp(-\lambda r(\theta))\,\mathrm{d}q(\theta)$, namely (1.1) is oscillatory if and only if $F(\lambda)\neq0$, for every $\lambda\in\mathbb{R}$. However, taking into account that for every $\lambda>0$

$$\left|\int_{-1}^{0}\exp(-\lambda r(\theta))\,\mathrm{d}q(\theta)\right|\le\exp(-\lambda m(r))\,\|q\|,$$

and that then $F(\lambda)\to+\infty$, as $\lambda\to+\infty$, we can conclude that (1.1) is oscillatory if only if

$$F(\lambda)>0,\qquad\forall\lambda\in\mathbb{R}. \tag{1.5}$$

2. Nonoscillations for continuous delays. In this section we will look for conditions in order to have (1.1) nonoscillatory. This happens whenever $F(\lambda) \leq 0$ for some $\lambda \in \mathbb{R}$.

Noticing that if $q(0) \geq 0$ then $F(0) = -q(0) \leq 0$, in such situation (1.1) is nonoscillatory independently of the delay functions $r \in C^{+}$. This happens, in particular, when q is nondecreasing on $[-1, 0]$.

A different situation for having (1.1) nonoscillatory is expressed in the following theorem.

Theorem 2.1. *Let $\theta_0 \in [-1, 0]$ be such that $r(\theta_0) = \|r\|$ and $r(\theta) < \|r\|$ for every $\theta \neq \theta_0$. If θ_0 is a point of increasing of $q(\theta)$, then equation (1.1) is nonoscillatory.*

Proof. For a matter of simplicity, let as assume that $\theta_0 = 0$.

Considering the decomposition of q given by (1.4),

$$q(\theta) = q_{+}(\theta) - q_{-}(\theta) \quad (\theta \in [-1, 0]),$$

$\theta_0 = 0$ will be a point of constancy of the function $q_{\_}$. Therefore we have for some $\varepsilon > 0$, sufficiently small, the function q_{+} increasing and q_{-} constant on $[-\varepsilon, 0]$, which means that

$$F(\lambda) = \lambda - \int_{-1}^{0} \exp(-\lambda r(\theta)) \, \mathrm{d}q_{+}(\theta) + \int_{-1}^{-\varepsilon} \exp(-\lambda r(\theta)) \, \mathrm{d}q_{-}(\theta).$$

Take $0 < \delta < \varepsilon$ in manner that $m_0 = \min\{r(\theta) : -\delta \leq \theta \leq 0\}$ be such that $m_0 > M = \max\{r(\theta) : -1 \leq \theta \leq -\varepsilon\}$. One easily can see that for every real $\lambda < 0$,

$$\begin{aligned} \int_{-1}^{0} \exp(-\lambda r(\theta)) \, \mathrm{d}q_{+}(\theta) &\geq \int_{-\varepsilon}^{0} \exp(-\lambda r(\theta)) \, \mathrm{d}q_{+}(\theta) \\ &\geq \int_{-\delta}^{0} \exp(-\lambda r(\theta)) \, \mathrm{d}q_{+}(\theta) \\ &\geq \exp(-\lambda m_0) (q_{+}(0) - q_{+}(-\delta)), \end{aligned}$$

and

$$0 \leq \int_{-1}^{-\varepsilon} \exp(-\lambda r(\theta)) \, \mathrm{d}q_{-}(\theta) \leq \exp(-\lambda M) \|q_{-}\|.$$

Thus, for every real $\lambda < 0$, we have

$$F(\lambda) \leq -\exp(-\lambda m_0)(q_{+}(0) - q_{+}(-\delta)) + \exp(-\lambda M) \|q_{-}\|,$$

that is,

$$F(\lambda) \leq -\exp(-\lambda m_0) \left[(q_{+}(0) - q_{+}(-\delta)) - \exp(-\lambda (m_0 - M)) \|q_{-}\| \right],$$

which shows that $F(\lambda) \to -\infty$, as $\lambda \to -\infty$. Hence (1.1) is nonoscillatory. □

Example 1. Consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \int_{-1}^{0} \cos(2\pi\theta) \, x(t - r(\theta)) \, \mathrm{d}\theta \tag{2.1}$$

where the delay function, $r \in C^+$, is strictly increasing in $[-1,0]$. With respect to (1.1), the corresponding function of NBV is $q(\theta) = \int_{-1}^{\theta} \cos(2\pi\theta)\,\mathrm{d}\theta$, which has a point of increase at $\theta = 0$, where $r(\theta)$ attains its absolute maximum. Hence (2.1) is nonoscillatory.

Corollary 2.2. *The equation* (1.2) *is nonoscillatory if* $a_k > 0$, *where the index* k *is determined by* $r_k = \max\{r_1, \ldots, r_p\}$.

Remark 1. We notice that the fact of the point θ_0, defined in the Theorem (2.1), be a point of increase of $q(\theta)$, has nothing to see with any increasing characteristics of this function in the whole interval $[-1,0]$. In fact, with $q(\theta)$ nondecreasing, or even increasing, on $[-1,0]$, the point θ_0 is not necessarily a point of increase of $q(\theta)$. Actually, in such case, θ_0 may eventually be a point of constancy of $q(\theta)$. Conversely,, if $q(\theta)$ has at θ_0 a point of increase, on the interval $[-1,0]$ it may not be a nondecreasing function. In fact, it may not be a monotonous function either.

When q is a decreasing function on $[-1,0]$ it cannot have at θ_0 a point of increase. Under that situation, in [4] are obtained several criteria in view of having (1.1) oscillatory, namely when

$$\int_{-1}^{0} r(\theta)\,\mathrm{d}q(\theta) < -\frac{1}{\mathrm{e}}. \tag{2.2}$$

However, in such case is possible to have a nonoscillatory situation as the stated in the following theorem.

Theorem 2.3. *If* $q(\theta)$ *is decreasing, then* (1.1) *is nonoscillatory if*

$$\|r\|\,|q(0)| \le \frac{1}{\mathrm{e}}. \tag{2.3}$$

Proof. If $\lambda < 0$, we have $\exp(-\lambda r(\theta)) \le \exp(-\lambda\|r\|)$, which implies that

$$-\int_{-1}^{0} \exp(-\lambda r(\theta))\,\mathrm{d}q(\theta) \le \exp(-\lambda\|r\|)\,|q(0)|.$$

Thus $F(\lambda) \le f(\lambda) = \lambda + \exp(-\lambda\|r\|)\,|q(0)|$. Since $f(\lambda) \to +\infty$, as $\lambda \to \pm\infty$, then $f(\lambda)$ has as absolute minimum the value $f(\lambda_0)$, for $\lambda_0 = \frac{1}{\|r\|}\log(\|r\|\,|q(0)|)$. But, by (2.3), $f(\lambda_0) = \frac{1}{\|r\|}[\log(\|r\|\,|q(0)|) + 1] \le 0$, which means that $F(\lambda_0) \le 0$. Hence (1.1) is nonoscillatory. □

Remark 2. If $q(\theta)$ is decreasing, then

$$\int_{-1}^{0} r(\theta)\,\mathrm{d}q(\theta) \ge \|r\|\,q(0) \ge -\frac{1}{\mathrm{e}}.$$

Therefore under (2.3) we are in the complement of condition (2.2).

Example 2. Let (1.1) for $q(\theta) = \frac{1}{2}\left(\theta^2 - 1\right)$ and $r(\theta) = \frac{(1-\mathrm{e})\theta + 1}{\mathrm{e}^3}$. As $q(0) = -\frac{1}{2}$ and $\|r\|\,|q(0)| = \frac{1}{2\,\mathrm{e}^2} < \frac{1}{\mathrm{e}}$. Therefore, by the Theorem 2.3, the corresponding equation (1.1) is nonoscillatory.

With respect to the equation (1.2), the Theorem 2.3, through the formulation (1.3), enables the following statement.

Corollary 2.4. *If* $a_j < 0$ *for* $j = 1, \ldots, p$, $r_1 < \ldots < r_p$, *and* $r_p \sum_{j=1}^{p} |a_j| \leq \frac{1}{\mathrm{e}}$, *then* (1.2) *is nonoscillatory.*

Example 3. By Corollary 2.4, one easily sees that the equation

$$\frac{d}{\mathrm{d}t} x(t) = -\frac{1}{8} x\left(t - \frac{1}{8}\right) - \frac{1}{8} x\left(t - \frac{1}{4}\right) - \frac{1}{4} x\left(t - \frac{1}{2}\right)$$

is nonoscillatory.

3. Oscillations and nonoscillations for differentiable delays. With $-1 \leq a \leq b \leq 0$, let $D^+(a,b)$ be the family of all functions in D^+ which are increasing on $[-1,a]$, constant on $[a,b]$ and decreasing on $[b,0]$. In case of having $a = b = \theta_0$ with $\theta_0 \in [-1,0]$ we obtain the family $D^+(\theta_0)$ of all differential and positive functions which are increasing on $[-1,\theta_0]$ and decreasing on $[\theta_0, 0]$. If $\theta_0 = -1$, $D^+(-1)$ is the class of all positive differentiable and decreasing functions on $[-1,0]$ wich we will denote by D_d^+. For $\theta_0 = 0$ we obtain the family D_i^+ of all positive differentiable and increasing funtions on $[-1,0]$. For these families of delays we start by stating the following oscillatory situation.

Theorem 3.1. *Let* $r \in D^+(a,b)$. *If*

$$\begin{aligned} &q(\theta) \geq 0 && \textit{for every } \theta \in [-1,a], \\ &q(\theta) \leq 0 && \textit{for every } \theta \in [b,0], \\ &q(0) < 0, \end{aligned} \tag{3.1}$$

and

$$\int_{-1}^{a} q(\theta)\, \mathrm{d}\log r(\theta) + \int_{b}^{0} q(\theta)\, \mathrm{d}\log r(\theta) < \frac{e}{r(0)} \left[\log\left(r(0)\,|q(0)|\right) + 1\right], \tag{3.2}$$

then (1.1) *is oscillatory.*

Proof. With $r \in D^+(a,b)$ we have

$$\begin{aligned} F(\lambda) = \lambda &- \int_{-1}^{a} \exp(-\lambda r(\theta))\, \mathrm{d}q(\theta) - \exp(-\lambda r(a))(q(b) - q(a)) \\ &- \int_{b}^{0} \exp(-\lambda r(\theta))\, \mathrm{d}q(\theta). \end{aligned}$$

Integrating by parts each one of the above integrals, we have

$$\begin{aligned} F(\lambda) = \lambda &+ \exp(-\lambda r(0))\,|q(0)| - \lambda \int_{-1}^{a} \exp(-\lambda r(\theta)) q(\theta)\, \mathrm{d}r(\theta) \\ &- \lambda \int_{b}^{0} \exp(-\lambda r(\theta))\, q(\theta)\, \mathrm{d}r(\theta). \end{aligned} \tag{3.3}$$

On the other hand, since $\lambda r(\theta) \exp(-\lambda r(\theta)) \leq \mathrm{e}^{-1}$, for every $\lambda \in \mathbb{R}$ and every $r(\theta)$, from (3.3) we obtain

$$\begin{aligned} F(\lambda) \ &\geq \lambda + \exp(-\lambda r(0))\,|q(0)| \\ &\quad - \mathrm{e}^{-1} \left[\int_{-1}^{a} q(\theta)\, \mathrm{d}\log r(\theta) + \int_{b}^{0} q(\theta)\, \mathrm{d}\log r(\theta)\right], \end{aligned}$$

for every real λ. As the function, $\varphi(\lambda) = \lambda + \exp(-\lambda r(0))|q(0)|$ similar to the function $f(\lambda)$ considered in the proof of THEOREM 2.3, is such that $\varphi(\lambda) \geq \frac{1}{r(0)}[\log(r(0) \cdot |q(0)|) + 1]$, for every $\lambda \in \mathbb{R}$, we conclude that

$$F(\lambda) \geq \frac{1}{r(0)}[\log(r(0)|q(0)|) + 1] - \mathrm{e}^{-1}\left[\int_{-1}^{a} q(\theta)\,\mathrm{d}\log r(\theta) + \int_{b}^{0} q(\theta)\,\mathrm{d}\log r(\theta)\right],$$

for every real λ. Thus, by (3.1), $F(\lambda) > 0$ for every real λ, and (1.1) is oscillatory. □

For the case where $a = b = \theta_0$, we first notice that it cannot be $\theta_0 = 0$. Otherwise (3.1) would be contradictory with respect to the value $q(0)$. For that case we have then the following corollary.

COROLLARY 3.2. *With* $\theta_0 \in [-1, 0[$ *let* $r \in D^+(\theta_0)$. *If*

$$\begin{aligned} &q(\theta) \geq 0 &&\textit{for every } \theta \in [-1, \theta_0] \\ &q(\theta) \leq 0 &&\textit{for every } \theta \in [\theta_0, 0], \\ &q(0) < 0, && \end{aligned} \tag{3.4}$$

and

$$\int_{-1}^{0} q(\theta)\,\mathrm{d}\log r(\theta) < \frac{\mathrm{e}}{r(0)}[\log(r(0)|q(0)|) + 1], \tag{3.5}$$

then (1.1) *is oscillatory.*

EXAMPLE 4. Let in (1.1)

$$q(\theta) = \begin{cases} 2(1+\theta), & \text{if } -1 \leq \theta \leq -\dfrac{1}{4}, \\ -1, & \text{if } -\dfrac{1}{4} \leq \theta \leq 0, \end{cases} \qquad r(\theta) = \begin{cases} -\theta^2 - \theta + \dfrac{3}{4}, & \text{if } -1 \leq \theta \leq -\dfrac{1}{2}, \\ 1, & \text{if } -\dfrac{1}{2} \leq \theta \leq -\dfrac{1}{4}, \\ -2\theta^2 - \theta + \dfrac{7}{8}, & \text{if } -\dfrac{1}{4} \leq \theta \leq 0. \end{cases}$$

As

$$\begin{aligned} &2\int_{-1}^{-\frac{1}{2}} (1+\theta)\,\mathrm{d}\log\left(-\theta^2 - \theta + \frac{3}{4}\right) - \int_{-\frac{1}{4}}^{0} \mathrm{d}\log\left(-2\theta^2 - \theta + \frac{7}{8}\right) \\ \approx\ &0.224 < \frac{8}{7}\mathrm{e}\left[\log\left[\frac{7}{8}\right] + 1\right] \approx 2.692, \end{aligned}$$

then, by THEOREM 3.1, (1.1) is oscillatory.

The special case $\theta_0 = -1$ gives the following corollary.

COROLLARY 3.3. *Let* $r \in D_d^+$. *If*

$$\begin{aligned} &q(\theta) \leq 0 \quad \textit{for every } \theta \in [-1, 0], \\ &q(0) < 0, \end{aligned}$$

and (3.5) *holds then* (1.1) *is oscillatory.*

Remark 3. The condition (2.2) and the condition (3.5) of Corollary 3.2 are independent in the following sense. If we consider functions r and q which are decreasing on $[-1,0]$, and such that $q(-1)=0$ and $q(0)<0$, that is, when the conditions (3.4) are satisfied for $\theta_0=-1$, then neither of two conditions (2.2) or (3.5) implies the other. Indeed in the following two examples we show that there are pairs of functions q, r, fulfilling the above conditions, which verify (2.2) but do not satisfy (3.5) (or verify (3.5) but do not satisfy (2.2)). In both examples the correspondent equation (1.1) is oscillatory.

Example 5. With $a>0$ let $q(\theta)=-\frac{1}{a}(\theta+1)$ and $r(\theta)=\frac{1}{5}-\theta$. Since $\int_{-1}^{0} r(\theta)\,\mathrm{d}q(\theta)=-\frac{7}{10a}$, the condition (2.2) is verified if and only if $a<\frac{7\,\mathrm{e}}{10}$. On the other hand, as

$$\int_{-1}^{0} q(\theta)\,\mathrm{d}\log r(\theta)=\frac{-5+6\log 6}{5a},$$

the condition (3.5) of Corollary 3.2 is satisfied if and only if $\frac{-5+6\log 6}{5a}>5\,\mathrm{e}\left(\log\frac{1}{5a}+1\right)$. One can see numerically that this inequality is satisfied if $a<0.4505$ and is not verified whenever $a\geq\frac{1}{2}$. Therefore for $\frac{1}{2}\leq a<\frac{7\,\mathrm{e}}{10}$ the condition (2.2) is verified, but (3.5) is not.

Example 6. As in the preceding example let $q(\theta)=-\frac{1}{a}(\theta+1)$, with $a>0$. Taking into account that for $r(\theta)=(1-\theta)^{1/2}$,

$$\int_{-1}^{0} r(\theta)\,\mathrm{d}q(\theta)=-\frac{1}{a}\left(-\frac{2}{3}+\frac{4\sqrt{2}}{3}\right),$$

the condition (2.2) is verified if and only if $a<\frac{\mathrm{e}}{3}\left(4\sqrt{2}-2\right)$. But in the regard of (3.5), this condition is satisfied if and only if $\int_{-1}^{0} q(\theta)\,\mathrm{d}\log r(\theta)=\frac{-1+2\log 2}{2a}>\mathrm{e}\left(\log\frac{2}{a}+1\right)$. Numerically one can see that this condition is not satisfied if $a\leq 5$. Thus for $\frac{\mathrm{e}}{3}\left(4\sqrt{2}-2\right)\leq a\leq 5$ the condition (3.5) is satisfied, but the condition (2.2) is not.

With respect to the equation (1.2) we can state the following corollary

Corollary 3.4. *If* $r_1>r_2>\ldots>r_p,\ a_j<0$, *for every* $j=1,\ldots,p$, *and*

$$\sum_{k=1}^{p-1}\left(\sum_{j=1}^{k} a_j\right)\log\frac{r_{k+1}}{r_k}<\frac{e}{r_p}\left[\log\left(r_p\left|\sum_{j=1}^{p} a_j\right|\right)+1\right]$$

then (1.2) *is oscillatory.*

For nonoscillations we state the following theorem.

Theorem 3.5. *Let* $r\in D^{+}(a,b)$ *and* $q\in NBV$ *such that*

$$\begin{aligned} &q(\theta)\geq 0 && \textit{for every } \theta\in[-1,a],\\ &q(\theta)\leq 0 && \textit{for every } \theta\in[b,0],\\ &-\frac{1}{\mathrm{e}\,r(0)}<q(0)<0. \end{aligned}$$

If

$$\int_{-1}^{a} q(\theta)\,\mathrm{d}r(\theta)+\int_{b}^{0} q(\theta)\,\mathrm{d}r(\theta)\leq\left(1+\frac{1}{\log\left(r(0)\left|q(0)\right|\right)}\right)\left(r(0)\left|q(0)\right|\right)^{\|r\|/r(0)} \quad (3.6)$$

then (1.1) *is nonoscillatory.*

Proof. Recalling (3.3),

$$F(\lambda) = \lambda + \exp(-\lambda r(0))|q(0)| \\ - \lambda\left[\int_{-1}^{a} \exp(-\lambda r(\theta))\, q(\theta)\,\mathrm{d}r(\theta) + \int_{b}^{0} \exp(-\lambda r(\theta))\, q(\theta)\,\mathrm{d}r(\theta)\right],$$

we have for $\lambda < 0$,

$$F(\lambda) \le \lambda + \exp(-\lambda r(0))|q(0)| \\ - \lambda \exp(-\lambda\|r\|)\left[\int_{-1}^{a} q(\theta)\,\mathrm{d}r(\theta) + \int_{b}^{0} q(\theta)\,\mathrm{d}r(\theta)\right]. \tag{3.7}$$

Recall also the function considered in the proof of the Theorem 3.1, $\varphi(\lambda) = \lambda + \exp(-\lambda r(0))|q(0)|$, and its absolute minimum $\varphi(\lambda_0) = \frac{1}{r(0)}[\log(r(0)|q(0)|) + 1]$, attained at $\lambda_0 = \frac{1}{r(0)}\log(r(0)|q(0)|)$.

Notice that since $r(0)|q(0)| < \mathrm{e}^{-1}$, one has $\lambda_0 < 0$. Therefore by (3.7) and (3.6) we obtain

$$F(\lambda_0) \le \varphi(\lambda_0) - \lambda_0 \exp(-\lambda_0\|r\|)\left(1 + \frac{1}{\log[r(0)|q(0)|]}\right)(r(0)|q(0)|)^{\|r\|/r(0)}.$$

Taking into account that

$$\lambda_0 \exp(-\lambda_0\|r\|) = \frac{(r(0)|q(0)|)^{-\|r\|/r(0)}}{r(0)}\log(r(0)|q(0)|)$$

we have then $F(\lambda_0) \le 0$ and (1.1) is nonoscillatory. □

Remark 4. Notice that the assumption $-\frac{\mathrm{e}^{-1}}{r(0)} < q(0) < 0$ implies

$$r(0)|q(0)| < \mathrm{e}^{-1}$$

and consequently that $\log(r(0)|q(0)|) + 1 < 0$. Thus in the Theorem 3.4 we are in the complementary of (3.2).

Example 7. Let

$$q(\theta) = \begin{cases} 1+\theta, & \text{if } -1 \le \theta \le -\frac{1}{2}, \\ -1, & \text{if } -\frac{1}{2} \le \theta \le 0, \end{cases} \qquad r(\theta) = \begin{cases} -\theta^2 - \frac{3}{2}\theta + \frac{51}{80}, & \text{if } -1 \le \theta \le -\frac{3}{4}, \\ \frac{6}{5}, & \text{if } -\frac{3}{4} \le \theta \le -\frac{1}{2}, \\ -\frac{4}{5}\theta^2 - \frac{4}{5}\theta + 1, & \text{if } -\frac{1}{2} \le \theta \le 0. \end{cases}$$

We have in this case

$$\begin{aligned} &-\frac{1}{2\,\mathrm{e}} < q(0) = -1 < 0, \\ &q(\theta) \ge 0, \qquad \text{for every } \theta \in \left[-1, -\tfrac{3}{4}\right] \\ &q(\theta) \le 0, \qquad \text{for every } \theta \in \left[-\tfrac{1}{2}, 0\right], \end{aligned}$$

and

$$\int_{-1}^{-\frac{3}{4}} (1+\theta)\, d\left(-\theta^2 - \frac{3}{2}\theta + \frac{51}{80}\right) + \int_{-\frac{3}{4}}^{0} q(0)\, d\left(-\frac{4}{5}\theta^2 - \frac{4}{5}\theta + 1\right)$$

$$\approx 0.0420 < \left(1 + \frac{1}{\log\frac{1}{2\,\mathrm{e}}}\right)\left(\frac{1}{2\,\mathrm{e}}\right)^{\frac{6}{5}} \approx 0.0537.$$

Then, by the THEOREM 3.5, (1.1) is nonoscilatory.

By setting in the THEOREM 3.5, $a = b = \theta_0$, the following corollary is obtained.

COROLLARY 3.6. *Let $r \in D^+(\theta_0)$ and $q \in NBV$ such that*

$$\begin{aligned} & q(\theta) \geq 0 && \text{for every } \theta \in [-1, \theta_0], \\ & q(\theta) \leq 0 && \text{for every } \theta \in [\theta_0, 0], \\ & -\frac{1}{\mathrm{e}\, r(0)} < q(0) < 0. \end{aligned}$$

If

$$\int_{-1}^{0} q(\theta)\,\mathrm{d}r(\theta) \leq \left(1 + \frac{1}{\log(r(0)|q(0)|)}\right)(r(0)|q(0)|)^{\|r\|/r(0)}$$

then (1.1) *is nonoscillatory.*

By choosing $\theta_0 = -1$, we obtain an important particular case of the COROLLARY 3.6.

COROLLARY 3.7. *Let $r \in D_d^+$ and $q \in NBV$ such that $q(\theta) \leq 0$, for $\theta \in [-1, 0[$ and $-\frac{1}{\mathrm{e}\, r(0)} < q(0) < 0$. If*

$$\int_{-1}^{0} q(\theta)\,\mathrm{d}r(\theta) \leq \left(1 + \frac{1}{\log(r(0)|q(0)|)}\right)(r(0)|q(0)|)^{\|r\|/r(0)},$$

then (1.1) *is nonoscillatory.*

With respect to equation (1.2) from (1.3) we can state the following corollary

COROLLARY 3.8. *If $r_1 > r_2 > \ldots > r_p$ and $-\frac{e^{-1}}{r(0)} < \sum_{j=1}^{p} a_j < 0$,*

$$\sum_{k=1}^{p-1}\left(\sum_{j=1}^{k} a_j\right)\frac{r_{k+1}}{r_k} \leq \left(1 + \frac{1}{\log\left(r_p\left|\sum_{j=1}^{p} a_j\right|\right)}\right)\left(r_p\left|\sum_{j=1}^{p} a_j\right|\right)$$

then (1.2) *is nonoscillatory.*

EXAMPLE 8. Let be $a_1 = -\frac{1}{8}, a_2 = -\frac{1}{4}$, $a_3 = -\frac{1}{8}$ and $r_1 = \frac{1}{2}$, $r_2 = \frac{1}{4}$, $r_3 = \frac{1}{8}$. Thus

$$\sum_{k=1}^{2}\left(\sum_{j=1}^{k} a_j\right)\log\frac{r_{k+1}}{r_k} = -\frac{1}{4}\log\frac{1}{2} \approx -0.173 < \left[\frac{1}{4} + \frac{1}{4\log\left(\frac{1}{4}\right)}\right] \approx 0.0696.$$

Therefore, by COROLLARY 3.8, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{8}x\left(t - \frac{1}{2}\right) - \frac{1}{4}x\left(t - \frac{1}{4}\right) - \frac{1}{8}x\left(t - \frac{1}{8}\right)$$

is nonoscillatory.

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