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ASYMPTOTIC EQUIVALENCE OF SYSTEMS OF DIFFERENCE EQUATIONS

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Abstract. The relation between sets of solutions of the first order linear system of difference equations and of its perturbation is studied. Asymptotic equivalence is proved using Tychonoff fixed point theorem.

Key words. System of difference equations, asymptotic equivalence, Tychonoff fixed point theorem.

AMS subject classifications. 39A11

Consider the system of difference equations

$$\Delta x_n = Ax_n + f(n, x_n), \tag{1}$$

where A is a $k \times k$ matrix, $k \in \mathbb{N}$, $x_n \in \mathbb{R}^k$, $n \in \mathbb{N}_0$ and $f\colon \mathbb{N}_0 \times \mathbb{R}^k \to \mathbb{R}^k$, $f(n, x)$ is continuous in x for any $n \in \mathbb{N}_0$ and Δ denotes the forward difference operator, i.e., $\Delta x_n = x_{n+1} - x_n$. Along with (1) we also consider the corresponding linear system

$$\Delta y_n = Ay_n. \tag{2}$$

Here $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In the sequel let us denote $\mathbb{N}(a) = \{a, a+1, \dots\}$ for $a \in \mathbb{N}_0$ and $\boldsymbol{x} = \{x_n\}$, $\boldsymbol{y} = \{y_n\}$, $\boldsymbol{z} = \{z_n\}$. Further, let us denote $|.|$ a norm on $\mathbb{R}^k$ or $\mathbb{R}^{k\times k}$ such that $|Dx| \le |D| \cdot |x|$ for any matrix D and any colon x.

In this contribution asymptotic relationship between solutions of the systems (1) and (2) will be investigated. The approach is inspired by the result for systems of ordinary differential equations studied in [8]. A similar topic for difference systems can be found in [4, 5, 7].

Definition 1. The systems (1) and (2) are said to be *asymptotically equivalent* if to each solution $\boldsymbol{x}$, $n \in \mathbb{N}(a)$, of (1) there exists a solution $\boldsymbol{y}$, $n \in \mathbb{N}(b)$, of (2) such that

$$\lim_{n\to\infty} |x_n - y_n| = 0 \tag{3}$$

and conversely to each solution $\boldsymbol{y}$, $n \in \mathbb{N}(a)$, of (2) there exists a solution $\boldsymbol{x}$, $n \in \mathbb{N}(b)$, of (1) such that (3) holds.

If (3) holds only for some subsets of all solutions of (1) and (2), we will speak about asymptotic equivalence between these sets.

First we will examine the special case of (1) – a non-homogeneous linear system

$$\Delta z_n = Az_n + b_n, \tag{4}$$

where $b_n \in \mathbb{R}^k$, $n \in \mathbb{N}_0$.

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We will suppose that the matrix $A + I = B$ is nonsingular, i.e., $\det B \neq 0$. This guarantees that each solution of (2) and (4) can be extended on $\mathbb{N}_0$. Denote Y_n, $n \in \mathbb{N}_0$, $Y_0 = I$, the fundamental matrix of (2). Thus $\Delta Y_n = AY_n$ or equivalently $Y_{n+1} = (A+I)Y_n$, from which we get $Y_n = (A+I)^n = B^n$, $n \in \mathbb{N}_0$.

As every solution z_n of (4) can be expressed like $z_n = y_n + \widehat{z}_n$, where $\widehat{z}_n$ is a fixed partial solution of (4) and y_n is an appropriate solution of (2), the proof of the next theorem is evident.

Theorem 1. *The systems* (2) *and* (4) *are asymptotically equivalent if and only if the system* (4) *possesses a solution* z_n *such that* $\lim_{n\to\infty} z_n = 0$.

Let us remind the variation of constant formula for (4). We look for a solution of (4) in the form $z_n = Y_n c_n$, where $c_n \in \mathbb{R}^k$ is an appropriate sequence. After substituting it to (4) we get

$$\begin{aligned} \Delta Y_n c_n + Y_{n+1}\Delta c_n &= AY_n c_n + b_n, \\ c_{n+1} &= c_n + Y_{n+1}^{-1} b_n, \end{aligned}$$

and therefore

$$c_n = c_0 + \sum_{i=1}^{n} Y_i^{-1} b_{i-1}.$$

Choosing $c_0 = 0$ we obtain that the system (4) has the solution

$$z_n = Y_n c_n = Y_n \sum_{i=1}^{n} Y_i^{-1} b_{i-1} = \sum_{i=1}^{n} B^{n-i} b_{i-1}. \tag{5}$$

If the series $\sum_{i=1}^{\infty} Y_i^{-1} b_{i-1}$ converges, then it is possible to adjust z_n:

$$z_n = Y_n \left(\sum_{i=1}^{\infty} Y_i^{-1} b_{i-1} - \sum_{i=n+1}^{\infty} Y_i^{-1} b_{i-1} \right) = Y_n \sum_{i=1}^{\infty} Y_i^{-1} b_{i-1} - Y_n \sum_{i=n+1}^{\infty} Y_i^{-1} b_{i-1}.$$

As $Y_n \sum_{i=1}^{\infty} Y_i^{-1} b_{i-1}$ is a solution of (2), we obtain that (4) has also a solution

$$z_n = -Y_n \sum_{i=n+1}^{\infty} Y_i^{-1} b_{i-1} = -\sum_{i=n+1}^{\infty} B^{n-i} b_{i-1}.$$

Assumption. Without loss of generality we can suppose that B has the Jordan canonical form.

Denote

$$(0 <)\ \mu_1 < \mu_2 < \cdots < \mu_s = \lambda$$

different absolute values of eigenvalues $\lambda_i(B)$, $i = 1, \ldots, t$. Let m_i be a maximal order of blocks that correspond to the eigenvalues with the absolute value μ_i. Let $m = m_s$. Further let us put

$$p = \begin{cases} m_j & \text{if } \mu_j = 1, \\ 1 & \text{if no } \mu_j \text{ equals } 1. \end{cases}$$

Assume $B = \operatorname{diag}(B_1, B_2)$, where

$$|\lambda_j(B_1)| \le \alpha = \max_j |\lambda_j(B_1)| < 1, \qquad m^\star = m_i \text{ if } \mu_i = \alpha,$$
$$|\lambda_j(B_2)| \ge 1 \qquad \text{for any } j.$$

Then $Y_n = (A+I)^n = B^n = \operatorname{diag}\{B_1^n, 0\} + \operatorname{diag}\{0, B_2^n\}$.

For $r \in \mathbb{N}$ consider a Jordan r-dimensional block

$$J = \begin{pmatrix} \mu & 1 & 0 & \dots & 0 \\ 0 & \mu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}.$$

For a sufficiently smooth function g defined at μ it can be defined (see [1])

$$g(J) = \begin{pmatrix} g(\mu) & g'(\mu) & \frac{g''(\mu)}{2!} & \dots & \frac{g^{(r-1)}(\mu)}{(r-1)!} \\ 0 & g(\mu) & g'(\mu) & \dots & \frac{g^{(r-2)}(\mu)}{(r-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & g(\mu) \end{pmatrix}.$$

Assume now that $\mu \ne 0$. Then especially for $g(\mu) = \mu^n$, $n \in \mathbb{Z}$ we have

$$J^n = \begin{pmatrix} \mu^n & n\mu^{n-1} & \dots & n(n-1)\cdots(n-r+2)\,\frac{\mu^{n-r+1}}{(r-1)!} \\ 0 & \mu^n & \dots & n(n-1)\cdots(n-r+1)\,\frac{\mu^{n-r+2}}{(r-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mu^n \end{pmatrix}. \tag{6}$$

From this it follows the existence of constants $K > 0$ and $L > 0$ such that

$$|B_1^n| \le K \cdot n^{m^\star - 1}\alpha^n \qquad \text{for } n \ge 1, \tag{7}$$
$$|B_2^{-n}| \le L \cdot (n+p-2)^{p-1} \qquad \text{for } n \ge 1. \tag{8}$$

Lemma 1. *Let $0 < q < 1$ and $g_n \ge 0$ for $n \in \mathbb{N}$. If*

$$\sum_{i=1}^{\infty} g_i < \infty,$$

then

$$\lim_{n\to\infty} q^n \sum_{i=1}^{n} q^{-i} g_i = 0.$$

Proof. Assume $\sum\limits_{i=1}^{\infty} g_i < \infty$. Denote $\lfloor x \rfloor$ an integer part of x, $x \in \mathbb{R}$. For any $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ there is

$$q^{n - \lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor} g_i < \frac{\varepsilon}{2} \qquad \text{and} \qquad \sum_{\lfloor n/2 \rfloor + 1}^{\infty} g_i < \frac{\varepsilon}{2}\,.$$

Then

$$q^n \sum_{i=1}^{n} q^{-i} g_i = q^n \sum_{i=1}^{\lfloor n/2 \rfloor} q^{-i} g_i + q^n \sum_{\lfloor n/2 \rfloor+1}^{n} q^{-i} g_i \le q^{n-\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor} g_i + \sum_{\lfloor n/2 \rfloor+1}^{n} g_i < \varepsilon. \qquad \square$$

Theorem 2. *Let*

$$\sum_{i=1}^{\infty} (i+p-2)^{p-1} |b_{i-1}| < \infty. \tag{9}$$

Then the equation (4) *has a solution* z_n *such that* $\lim_{n\to\infty} z_n = 0$.

Proof. Denote $\widehat{B}_1 = \operatorname{diag}\{B_1, O_2\}$, $\widehat{B}_2 = \operatorname{diag}\{O_1, B_2\}$, $\widehat{B}_1^{-1} = \operatorname{diag}\{B_1^{-1}, O_2\}$ and $\widehat{B}_2^{-1} = \operatorname{diag}\{O_1, B_2^{-1}\}$, where O_i is a zero matrix of the same dimension as B_i, $i = 1, 2$. Further $Y_n = \widehat{B}_1^n + \widehat{B}_2^n$ holds.

From (5) we see that any solution z_n of (4) can be written in the form

$$z_n = Y_n z_0 + \sum_{i=1}^{n} B^{n-i} b_{i-1}.$$

Therefore

$$z_n = Y_n z_0 + \widehat{B}_1^n \sum_{i=1}^{n} \widehat{B}_1^{-i} b_{i-1} + \widehat{B}_2^n \sum_{i=1}^{n} \widehat{B}_2^{-i} b_{i-1}.$$

The inequalities (8) and (9) imply that the series $\sum\limits_{i=1}^{\infty} \widehat{B}_2^{-i} b_{i-1}$ absolutely converges, so we can express z_n as follows:

$$z_n = \widehat{B}_1^n z_0 + \widehat{B}_2^n \left[z_0 + \sum_{i=1}^{\infty} \widehat{B}_2^{-i} b_{i-1} \right] + \widehat{B}_1^n \sum_{i=1}^{n} \widehat{B}_1^{-i} b_{i-1} - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} b_{i-1}.$$

Choose $z_0 + \sum\limits_{i=1}^{\infty} \widehat{B}_2^{-i} b_{i-1} = 0$. Then

$$z_n = \widehat{B}_1^n z_0 + \widehat{B}_1^n \sum_{i=1}^{n} \widehat{B}_1^{-i} b_{i-1} - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} b_{i-1} = I_1 + I_2 + I_3. \tag{10}$$

We will show that $\lim\limits_{n\to\infty} I_i = 0$, $i = 1, 2, 3$.

From (7) evidently $\lim\limits_{n\to\infty} I_1 = 0$.

Further we estimate I_2. We have

$$|I_2| \le \left| \sum_{i=1}^{n-1} \widehat{B}_1^{n-i} b_{i-1} \right| + |\widehat{B}_1^0 b_{n-1}| = |I_{2a}| + |I_{2b}|.$$

If $1 \le i \le n-1$, then $n - i \ge 1$ and from (7) we have

$$|\widehat{B}_1^{n-i} b_{i-1}| \le K(n-i)^{m^\star - 1} \alpha^{n-i} |b_{i-1}|.$$

As $\lim\limits_{k\to\infty} k^{m^\star-1}(\sqrt{\alpha})^k = 0$, because $\sqrt{\alpha} < 1$, a constant $M > 0$ exists such that

$$0 \le (n-i)^{m^\star-1}(\sqrt{\alpha})^{n-i} \le M$$

for $n \ge i$. Therefore

$$\begin{aligned}|I_{2a}| &\le \sum_{i=1}^{n-1} K(n-i)^{m^\star-1}\alpha^{n-i}|b_{i-1}| \le KM\sum_{i=1}^{n-1}(\sqrt{\alpha})^{n-i}|b_{i-1}|\\ &= KM(\sqrt{\alpha})^n\sum_{i=1}^{n}(\sqrt{\alpha})^{-i}|b_{i-1}| \to 0\end{aligned}$$

for $n \to \infty$ by Lemma 1.

Due to (9) we have $b_{n-1} \to \infty$ for $n \to \infty$ and

$$|I_{2b}| \le |\widehat{B}_1^0| \cdot |b_{n-1}| \to \infty$$

for $n \to \infty$. Thus $\lim\limits_{n\to\infty} I_2 = 0$.

Now we estimate I_3. From (8) we obtain

$$\begin{aligned}|I_3| &\le \sum_{i=n+1}^{\infty}|\widehat{B}_2^{-(i-n)}| \cdot |b_{i-1}| \le \sum_{i=n+1}^{\infty} L(i-n+p-2)^{p-1}|b_{i-1}|\\ &\le L\sum_{i=n+1}^{\infty}(i+p-2)^{p-1}|b_{i-1}| \to 0\end{aligned}$$

for $n \to \infty$ by (9). Thus $\lim\limits_{n\to\infty} I_3 = 0$. □

Consider a linear space $\ell(n_1)$ of real sequences $\boldsymbol{x} = \{x_n\}$, $x_n \in \mathbb{R}^k$, $n \ge n_1$, $n_1 \in \mathbb{N}_0$, endowed with the topology τ induced by the set of seminorms $P_m(x) = |x_m|$, $m \ge n_1$. Then $(\ell(n_1), \tau)$ is a Fréchet space – see [6, p. 37]. If $\boldsymbol{x}^r = \{x_n^r\} \in \ell(n_1)$, $r \in \mathbb{N}_0$, then $\boldsymbol{x}^r \to \boldsymbol{x}^0$ in $(\ell(n_1), \tau)$ if and only if $\lim\limits_{r\to\infty} x_n^r = x_n^0$ for each $n \ge n_1$.

Lemma 2. *A subset $B \subset \ell(n_1)$ is relatively compact in $(\ell(n_1), \tau)$ if and only if $\sup\limits_{\boldsymbol{x}\in B}\{|x_n|\} < \infty$ for each $n \ge n_1$.*

Proof. Sufficiency: Consider a sequence $\{\boldsymbol{x}^r\} \subset B$. It is possible to choose its subsequence $\{\boldsymbol{y}^r\}$ such that $\{y_1^r\}$ is Cauchy. Then it is possible to choose a subsequence $\{\boldsymbol{z}^r\}$ of $\{\boldsymbol{y}^r\}$ such that $\{z_2^r\}$ is Cauchy etc. The diagonal sequence $\{\boldsymbol{y}^1, \boldsymbol{z}^2, \dots\}$ is chosen from $\{\boldsymbol{x}^r\}$ and is Cauchy in $(\ell(n_1), \tau)$.

Necessity: If $\sup\limits_{\boldsymbol{x}\in B}\{|x_n|\} = \infty$ for some $n \ge n_1$, it is possible to find a sequence $\{\boldsymbol{x}^r\} \subset B$ such that $\lim\limits_{r\to\infty}|x_n^r| = \infty$. Evidently, for no subsequence $\{\boldsymbol{y}^r\}$ of $\{\boldsymbol{x}^r\}$ the sequence $\{|y_n^r|\}$ is Cauchy. □

Theorem 3. *Assume*

$$|f(n,x)| \le F(n,|x|), \tag{11}$$

where $F\colon \mathbb{N}_0 \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $F(n,u)$ is nondecreasing and continuous in u for any $n \in \mathbb{N}_0$. Let

$$\sum_{i=1}^{\infty}(i+p-2)^{p-1}F(i-1,c) < \infty \tag{12}$$

for any $c \in \mathbb{R}_0^+$.

Then the sets of bounded solutions of (1) and (2) are asymptotically equivalent.

Proof. Let x_n, $n \in \mathbb{N}(n_0)$, be a bounded solution of (1). Then there exists $c \geq 0$ such that $|x_n| \leq c$ for $n \in \mathbb{N}(n_0)$. If y_n is an arbitrary solution of (2), then $z_n = x_n - y_n$ is a solution of an equation

$$\Delta z_n = A z_n + f(n, x_n). \tag{13}$$

Conversely, if z_n is an arbitrary solution of (13), then $y_n = x_n - z_n$ is a solution of (2).

According to (11) and (12) we have

$$\begin{aligned}\sum_{i=1}^{\infty}(i+p-2)^{p-1}|f(i-1,x_{i-1})| &\leq \sum_{i=1}^{\infty}(i+p-2)^{p-1}F(i-1,|x_{i-1}|)\\ &\leq \sum_{i=1}^{\infty}(i+p-2)^{p-1}F(i-1,c) < \infty.\end{aligned}$$

Then due to Theorem 2 the equation (13) has a solution z_n such that $\lim_{n\to\infty} z_n = 0$. Thus $y_n = x_n - z_n$ is a solution of (2) for which $\lim_{n\to\infty}|x_n - y_n| = \lim_{n\to\infty}|z_n| = 0$.

Assume now that y_n is a bounded solution of (2), $n \in \mathbb{N}_0$. Consider an equation

$$x_n = y_n + \widehat{B}_1^n \sum_{i=n_0+1}^{n} \widehat{B}_1^{-i} f(i-1, x_{i-1}) - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1}), \tag{14}$$

$n_0 \in \mathbb{N}_0$, $n \geq n_0$. It is easy to verify that any solution of (14) is a solution of (1), too. In fact, if we denote $I = \widehat{I}_1 + \widehat{I}_2 = \operatorname{diag}\{I_1, O_2\} + \operatorname{diag}\{O_1, I_2\}$, where I_i is a unit matrix of the same dimension as B_i, $i = 1, 2$, we obtain

$$\begin{aligned}\Delta x_n = \Delta y_n &+ \Delta\widehat{B}_1^n \sum_{i=n_0+1}^{n} \widehat{B}_1^{-i} f(i-1,x_{i-1}) + \widehat{B}_1^{n+1}\widehat{B}_1^{-(n+1)} f(n,x_n)\\ &- \Delta\widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1,x_{i-1}) + \widehat{B}_2^{n+1}\widehat{B}_2^{-(n+1)} f(n,x_n)\\ = A y_n &+ A\widehat{B}_1^n \sum_{i=n_0+1}^{n} \widehat{B}_1^{-i} f(i-1,x_{i-1}) + \widehat{I}_1 f(n,x_n)\\ &- A\widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1,x_{i-1}) + \widehat{I}_2 f(n,x_n)\\ = A x_n &+ f(n,x_n).\end{aligned}$$

For $\rho > 0$, let us denote $\mathscr{B}_\rho = \{\boldsymbol{x} \in \ell(n_0) : |x_n| \leq \rho, n \in \mathbb{N}(n_0)\}$. Evidently $\mathscr{B}_\rho$ is a convex closed subset of $\ell(n_0)$, which is compact by Lemma 2. We will show that (14) has a solution in $\mathscr{B}_\rho$, where ρ and n_0 will be chosen later.

Define a mapping $T \colon \mathscr{B}_\rho \to \ell(n_0)$ as follows:

$$T(\boldsymbol{x})_n = y_n + \widehat{B}_1^n \sum_{i=n_0+1}^{n} \widehat{B}_1^{-i} f(i-1, x_{i-1}) - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1})$$

for $\boldsymbol{x} \in \mathscr{B}_\rho$. By (8) and (11) we have

$$|\widehat{B}_2^{-i} f(i-1,x_{i-1})| \leq L(i+p-2)^{p-1}F(i-1,|x_{i-1}|) \leq L(i+p-2)^{p-1}F(i-1,\rho),$$

which shows (using (12)) that $\sum\limits_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1})$ converges and T is correctly defined.

Let $|y_n| \le c_1$, $n \in \mathbb{N}_0$, and choose $\rho \ge 2c_1$. Then

$$|T(\boldsymbol{x})_n| \le |y_n| + \sum_{i=n_0+1}^{n} |\widehat{B}_1^{n-i}| F(i-1, \rho) + \sum_{i=n+1}^{\infty} |\widehat{B}_2^{-(i-n)}| F(i-1, \rho).$$

With respect to (7) there exists a constant $M > 0$ such that $|\widehat{B}_1^{n-i}| \le M$ for $n_0+1 \le i \le n$. Using (12) we get that $\sum\limits_{i=1}^{\infty} F(i-1, \rho) < \infty$, which implies that

$$\begin{aligned} |T(\boldsymbol{x})_n| &\le c_1 + M \sum_{i=n_0+1}^{n} F(i-1, \rho) + L \sum_{i=n+1}^{\infty} (i-n+p-2)^{p-1} F(i-1, \rho) \\ &\le c_1 + M \sum_{i=n_0+1}^{\infty} F(i-1, \rho) + L \sum_{i=n+1}^{\infty} (i+p-2)^{p-1} F(i-1, \rho). \end{aligned}$$

Therefore, it is possible to find n_0 large enough such that

$$M \sum_{i=n_0+1}^{\infty} F(i-1, \rho) + L \sum_{i=n+1}^{\infty} (i+p-2)^{p-1} F(i-1, \rho) < c_1.$$

Then $|T(\boldsymbol{x})_n| \le 2c_1 \le \rho$. i.e. $T\colon \mathscr{B}_\rho \to \mathscr{B}_\rho$.

Further we will verify that T is a continuous operator. Let $\boldsymbol{x}^k \to \boldsymbol{x}$ for $k \to \infty$ in $\ell(n_0)$, which means that $\lim\limits_{k\to\infty} x_n^k = x_n$, $n \in \mathbb{N}(n_0)$. Choose a fixed $n \ge n_0$ and $n_1 > n$. Then

$$\begin{aligned} |T(\boldsymbol{x}^k)_n - T(\boldsymbol{x})_n| &= \Big| \widehat{B}_1^{n} \sum_{i=n_0+1}^{n} \widehat{B}_1^{-i} [f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})] \\ &\qquad - \widehat{B}_2^{n} \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} [f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})] \Big| \\ &\le \sum_{i=n_0+1}^{n} |\widehat{B}_1^{n-i}| \cdot |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\qquad + \sum_{i=n+1}^{n_1} |\widehat{B}_2^{-(i-n)}| \cdot |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\qquad + 2 \sum_{i=n_1+1}^{\infty} |\widehat{B}_2^{-(i-n)}| F(i-1, \rho) \\ &\le M \sum_{i=n_0+1}^{n_1} |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\qquad + L \sum_{i=n+1}^{n_1} (i-n+p-2)^{p-1} |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\qquad + 2 \sum_{i=n_1+1}^{\infty} (i-n+p-2)^{p-1} F(i-1, \rho) \end{aligned}$$

$$\begin{aligned}&\le M\sum_{i=n_0+1}^{n_1}|f(i-1,x_{i-1}^k)-f(i-1,x_{i-1})|\\&\quad+L(n_1+p-2)^{p-1}\sum_{i=n_0+1}^{n_1}|f(i-1,x_{i-1}^k)-f(i-1,x_{i-1})|\\&\quad+2\sum_{i=n_1+1}^{\infty}(i+p-2)^{p-1}F(i-1,\rho).\end{aligned}$$

Let $\varepsilon>0$ be an arbitrary number. From (12) there exists n_1 such that

$$\sum_{i=n_1+1}^{\infty}(i+p-2)^{p-1}F(i-1,\rho)<\frac{\varepsilon}{4}\,.$$

Further, as $\boldsymbol{x}^k\to\boldsymbol{x}$ in $\ell(n_0)$ and f is continuous at its second argument we can find $k_0\in\mathbb{N}$ such that for $k\ge k_0$ and $n_0\le i\le n_1-1$ the inequality

$$|f(i,x_i^k)-f(i,x_i)|<\frac{\varepsilon}{2(n_1-n_0)\bigl(M+L(n_1+p-2)^{p-1}\bigr)}$$

holds. Thus $|T(\boldsymbol{x}^k)_n-T(\boldsymbol{x})_n|<\varepsilon$ for $k\ge k_0$, i.e. $\lim\limits_{k\to\infty}T(\boldsymbol{x}^k)_n=T(\boldsymbol{x})_n$. But this means that $T(\boldsymbol{x}^k)\to T(\boldsymbol{x})$ as $k\to\infty$ in $\ell(n_0)$.

From Tychonoff fixed point theorem – see [2, p. 405] or [3, p. 45] – we conclude that T has a fixed point in $\mathscr{B}_\rho$. Thus (14) and also (1) has a bounded solution.

We will show that $\lim\limits_{n\to\infty}|x_n-y_n|=0$. Analogously as in the proof of Theorem 2 for I_2 and I_3 (see (10)) it can be proved that

$$\sum_{i=n_0+1}^{n}\widehat{B}_1^{n-i}f(i-1,x_{i-1})-\sum_{i=n+1}^{\infty}\widehat{B}_2^{-(i-n)}f(i-1,x_{i-1})\to 0$$

as $n\to\infty$ (b_{i-1} is to be replaced by $F(i-1,\rho)$). This proves the theorem. □

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