Denis Bonheure; Christian Fabry; Didier Smets

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On a class of forced nonlinear oscillators at resonance

Denis Bonheure[1], Christian Fabry[2] and Didier Smets[3]

[1] Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée,
Chemin du Cyclotron, 2 , B-1348 Louvain-la-Neuve, Belgium
Email: bonheure@anma.ucl.ac.be

[2] Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée,
Chemin du Cyclotron, 2 , B-1348 Louvain-la-Neuve, Belgium
Email: Fabry@amm.ucl.ac.be

[3] Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée,
Chemin du Cyclotron, 2 , B-1348 Louvain-la-Neuve, Belgium
Email: Smets@amm.ucl.ac.be

Abstract. We present existence, non-existence and multiplicity results for periodic solutions of forced nonlinear oscillators at resonance, the nonlinearity being a bounded perturbation of a force deriving from an isochronous potential, i.e. a potential leading to free oscillations that all have the same period. The class of nonlinearities considered includes jumping nonlinearities, as well as singular forces of repulsive type. As particular cases of the existence results, we obtain conditions of Landesman-Lazer type. We also investigate the problem of boundedness of the solutions.

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1 Introduction

In this note, we summarize some results obtained recently in [2] and [3]. We refer the reader to these papers for a more detailed exposition.

We consider the following class of nonlinear oscillators

$$x'' + V'(x) + g(x) = p(t), \tag{1}$$

in situations of resonance. The following hypotheses, where $k \in \mathbb{N}^*$ and $a \in [-\infty, 0)$, are assumed to hold throughout:

This is an overview article.

($\mathcal{H}_k$) $V:(a,+\infty)\to\mathbb{R}$ is a $2\pi/k$-isochronous, strictly convex potential whose derivative is locally lipschitzian; $g:(a,+\infty)\to\mathbb{R}$ is bounded and locally lipschitzian; p belongs to $L^1_{loc}(\mathbb{R})$ and is 2π-periodic.

By convention, we suppose that the minimum of V is reached at 0, so that $V'(0)=0$. By a $2\pi/k$-isochronous potential, we mean that all nontrivial solutions of

$$x''+V'(x)=0$$

are of (minimal) period $2\pi/k$. We also assume that V satisfies either

$$(\mathcal{S})\qquad \lim_{x\to+\infty}\frac{V'(x)}{x}=\frac{k^2}{4}\quad\text{and}\quad \lim_{x\to a_+}\frac{V'(x)}{x}=+\infty,$$

where $a\in[-\infty,0)$, or

$$(\mathcal{NS})\qquad \lim_{x\to+\infty}\frac{V'(x)}{x}=\alpha>0\quad\text{and}\quad \lim_{x\to-\infty}\frac{V'(x)}{x}=\beta>0$$

(in which case $a=-\infty$).

The first case is referred to as the singular case, because when $a\neq-\infty$, it corresponds to a repulsive singularity which, by convention, has been placed here on the negative side. In the second case, which is referred to as the non-singular case, V' is asymptotic to a so-called jumping or asymmetric nonlinearity $\alpha x^+-\beta x^-$, where $x^+=\max\{x,0\}, x^-=\max\{-x,0\}$. The isochronism assumption implies that

$$\frac{1}{\sqrt{\alpha}}+\frac{1}{\sqrt{\beta}}=\frac{2}{k}. \tag{2}$$

We refer to [1] and [3] for examples of isochronous potentials. It is shown in [3] (see also Corollary 4 below) that perturbations of functions deriving from an isochronous potential cover a large class of nonlinearities.

Consider equation (1) where V,g,p satisfy hypothesis ($\mathcal{H}_k$) for some $k\in\mathbb{N}^*$, and V satisfies either ($\mathcal{S}$) or ($\mathcal{NS}$). Let us define the function Φ by

$$\Phi(\theta)=\int_0^{2\pi}p(t)\psi(t+\theta)\,dt\,, \tag{3}$$

where ψ denotes either $|\cos(kt/2)|$ in case ($\mathcal{S}$) holds, or the solution of

$$x''+\alpha x^+-\beta x^-=0, x(0)=1, x'(0)=0,$$

when ($\mathcal{NS}$) holds. Notice that Φ, as ψ, is of period $2\pi/k$. Let us also define

$$G(\rho)=\int_0^{2\pi}g(\rho\psi(t))\psi(t)\,dt\,,$$

and its limits

$$G_+=\liminf_{\rho\to+\infty}G(\rho)\quad\text{and}\quad G^+=\limsup_{\rho\to+\infty}G(\rho)\,.$$

We show in [2], [3] and [4] that the function Φ plays a key role in the problem of existence of 2π-periodic solutions of (1) as well as in the problem of boundedness of the solutions of equation (1).

2 Periodic solutions

Concerning the existence of periodic solutions, we prove in [3] the following theorem.

Theorem 1. *Let Φ, G_+, G^+ be defined as above. We have the following :*

(i) *If there exists $G^* \in [G_+, G^+]$, which is a regular value of Φ, and if the number of zeros of $\Phi - G^*$ in $[0, 2\pi/k)$ is different from 2, equation (1) has at least one 2π-periodic solution.*

(ii) *If there exist two regular values $G_1, G_2 \in [G_+, G^+]$ of Φ such that the number of zeros of $\Phi - G_1$ and $\Phi - G_2$ in $[0, 2\pi/k)$ are different, equation (1) admits an unbounded sequence of 2π-periodic solutions.*

(iii) *If $[G_+, G^+]$ contains no critical value of Φ, the set of 2π-periodic solutions of (1) is bounded.*

The existence condition includes the case of a function $\Phi - G^*$ of constant sign. In particular, the result applies if

$$\max \Phi < G^+ \quad \text{or} \quad \min \Phi > G_+ \,. \tag{4}$$

That situation has been treated by Krasnosel'skii and Mawhin [9] for perturbations of a linear oscillator.

Arguing as in [9], it can be shown that, when g has a sublinear primitive, $G_+ = G^+ = 0$. Hence, if one considers, for instance, the equation

$$x'' + \alpha x^+ - \beta x^- + \sin(x) = p(t) \,,$$

where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}$, and α, β satisfy condition (2) with $k = 1$, or the equation

$$x'' - \frac{1}{4(x+1)^3} + \frac{(x+1)}{4} + \sin(x) = p(t) \,,$$

for both of which $g(x) = \sin(x)$, it results from Theorem 1 that these equations have at least one 2π-periodic solution if the number of zeros (supposed to be simple), in $[0, 2\pi)$, of the function Φ defined by (3), is different from 2.

Several earlier results can be obtained as particular cases of the above theorem, corresponding to situations where the image of Φ does not intersect $[G_+, G^+]$. Noting that

$$G^+ \leq \left(\limsup_{x \to +\infty} g(x)\right) \int_{\psi > 0} \psi + \left(\liminf_{x \to -\infty} g(x)\right) \int_{\psi < 0} \psi \,,$$

and writing a similar inequality for G_+, the following corollary is obtained, after computation of the integrals for ψ.

Corollary 2. *Suppose that*

$$\frac{\Phi(\theta)}{2\sqrt{\alpha}} > \left(\limsup_{x\to+\infty} g(x)\right)\frac{k}{\alpha} - \left(\liminf_{x\to-\infty} g(x)\right)\frac{k}{\beta}, \text{ for all } \theta \in [0, 2\pi],$$

or

$$\frac{\Phi(\theta)}{2\sqrt{\alpha}} < \left(\liminf_{x\to+\infty} g(x)\right)\frac{k}{\alpha} - \left(\limsup_{x\to-\infty} g(x)\right)\frac{k}{\beta}, \text{ for all } \theta \in [0, 2\pi].$$

Then equation (1) has at least one 2π-periodic solution. Notice that β has to be considered as $+\infty$ in the singular case, α being then equal to $k^2/4$.

The conditions appearing in the above corollary are conditions of Landesman-Lazer type; they are clearly more restrictive than (4).

On the other hand, if g admits limits $g(\pm\infty)$ at $\pm\infty$, it is immediate that

$$G_+ = G^+ = 2k\sqrt{\alpha}\left(\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\right), \tag{5}$$

still with $\beta = +\infty$ in the singular case, so that the following corollary can be stated.

Corollary 3. *Assume that g admits limits at $\pm\infty$. If $G_+ = G^+$, given by (5), is a regular value of Φ and if the number of zeros of $\Phi - G_+$ in $[0, 2\pi/k)$ is different from 2, equation (1) has at least one 2π-periodic solution.*

The situation of the above corollary has been treated by Fabry-Fonda [7] for equations with jumping nonlinearities. When g admits limits at $\pm\infty$, Corollary 3 provides existence conditions that are more general than conditions of Landesman-Lazer type.

Whether the Fredholm type conditions of Theorem 1 are necessary is probably false; on the other hand, existence of solutions for whatever p is also false. We prove in [3] that, for ε sufficiently small, $\varepsilon \neq 0$, the equation

$$x'' + V'(x) = \varepsilon \sin t \tag{6}$$

has no 2π-periodic solutions, assuming that V satisfies hypothesis $(\mathcal{H}_k)$ with $k = 1$, and that $V'(x)$ satisfies either $(\mathcal{S})$ or $(\mathcal{NS})$. We assume moreover that V' admits a derivative at 0.

It does not seem easy to give simple conditions ensuring that a nonlinearity falls into the scope of the preceding theorems. Given an equation

$$x'' + q(x) = p(t),$$

it is not clear if the existence conditions can be easily stated in terms of q and p. In the singular case, we are able to prove the following. We denote by Q the primitive of q which is zero at zero.

Corollary 4. *Assume that p is locally integrable and 2π-periodic, q is locally lipschitzian, $\lim_{x\to -a} Q(x) = +\infty$ and $\lim_{x\to+\infty}(q(x) - x/4) = g_\infty/4$. Moreover, assume that there exists $\delta > 0$ such that for every $x \in (-a, -a+\delta)$:*

(i) $q'(x) > 0$,

(ii) $|q'(x)| < Q(x)^{-3/2}|q(x)|^3/\sqrt{2}$.

Then, if the number of zeros of $\Phi - (g_\infty - a)$ in $[0, 2\pi)$ is not equal to 2, those zeros being simple, equation

$$x'' + q(x) = p(t)$$

admits at least one 2π-periodic solution.

Corollary 4 provides an answer to a question raised by Del Pino and al. (Remark 1.2 in [6]). For the model equation

$$x'' - \frac{1}{x^\nu} + \beta x = p(t),$$

with $\nu \geq 1$, Del Pino, Manásevich and Montero proved the existence of at least one 2π-periodic solution for $\beta \neq k^2/4$ and p continuous 2π-periodic. Using a shift in the x coordinate, Corollary 4 can be applied to this model equation. Moreover we can show that for this model, $g_\infty = a$. It gives then an explicit existence condition for 2π-periodic solutions in the resonant cases. On the other hand, as mentioned above, in the resonant cases a non-existence result is proved in [3]. For example, the equation

$$x'' - \frac{1}{x^3} + \frac{1}{4}x = \varepsilon \sin t$$

has no 2π-periodic solution, at least for ε small.

3 Unbounded solutions

For the simpler equation

$$x'' + \alpha x^+ - \beta x^- = p(t), \tag{7}$$

with p smooth, it has been shown by Liu in [10] that all the solutions are bounded if Φ is of constant sign. This is also true for nonlinearities deriving from regular isochronous potentials, as proved in [4].

By contrast, it follows from results of Fabry and Mawhin [8] that, if the function Φ has zeros, all being simple, then the large amplitude solutions of (7) are unbounded either in the past or in the future (see also [5]). On the other hand, we show in [3] that the equation (6) where V is a smooth potential satisfying hypothesis $(\mathcal{H}_k)$ with $k = 1$, and either $(\mathcal{S})$ or $(\mathcal{NS})$, has no 2π-periodic solution, at least for ε small. By a result of Massera [11], all the solutions of equation (6) are then unbounded for ε small. It is easy to check that for the forcing term $p(t) = \sin t$, the function Φ has exactly two (simple) zeros in $[0, 2\pi)$.

All the above results suggest that, for equation (1), the boundedness of the solutions depends on whether Φ vanishes at some point or not, at least when $G^+ = G_+ = 0$. We show in [2] how to adapt the condition to the general case of arbitrary values of G^+ and G_+. We prove the following result, which improves Theorem 2 of [8].

Theorem 5. *Let Φ, G_+, G^+ be defined as above. Suppose that* $\max \Phi > G^+$*,* $\min \Phi < G_+$ *and that $[G_+, G^+]$ does not contain any critical value of Φ. Then, there exists $R > 0$ such that all the solutions $x(t)$ of (1) satisfying $(x(0))^2 + (x'(0))^2 > R$ are unbounded, either in the future or in the past.*

Notice that Theorem 5 does apply to the particular case $G_+ = G^+$, which holds for a large class of functions g. When $G^+ = G_+$, the assumption of Theorem 5 amounts to ask that $\Phi - G^+$ vanishes at some point, the zeros being simple. Hence, if one considers, for instance, the equation

$$x'' + \alpha x^+ - \beta x^- + \sin(x) = p(t)\,, \tag{8}$$

where α, β satisfy (2) with $k = 1$, or the equation

$$x'' - \frac{1}{4(x+1)^3} + \frac{(x+1)}{4} + \sin(x) = p(t)\,,$$

for both of which $g(x) = \sin(x)$, it results from Theorem 5 that if the function Φ defined by (3) vanishes at some point in $[0, 2\pi)$ (the zeros are supposed to be simple), then the large amplitude solutions of these equations are unbounded either in the past or in the future. In the particular case where $k = 1$ and $p(t) = a + b\cos t$, it is easily computed that unbounded solutions are present when $3|a| < |b|$. Equation (8) is already covered by the results of [8] but, when $G^+ \neq G_+$, the result of Theorem 5 is new, even in the case of a harmonic oscillator. We refer to Proposition 1 of [3] for examples where the limits G^+ and G_+ are different and where these can be easily computed from the limits of g.

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