Pavel Šolín; Karel Segeth

Non-uniqueness of solution to quasi-1D compressible Euler equations

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 379--389.

Persistent URL: http://dml.cz/dmlcz/700370

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 379–389

Non-Uniqueness of Solution to Quasi-1D Compressible Euler Equations

Pavel Šolín[1] and Karel Segeth[2]⋆

[1] Institute of Industrial Mathematics, Johannes Kepler University,
Altenbergerstrasse 69, A-4040 Linz, Austria
Email: solin@indmath.uni-linz.ac.at
[2] Mathematical Institute of the Academy of Sciences,
Žitná 25, 11567 Prague, Czech Republic
Email: segeth@matsrv.math.cas.cz

Abstract. Our study deals with the non-unique behaviour of the almost-unidirectional flow of compressible inviscid gases. This phenomenon is not unknown to engineers who deal with compressible flows in pipes, ducts, tubes or nozzles of nontrivial geometries, however it is not easy to find an exact mathematical analysis in the engineering literature. Our aim is to fill this gap by providing a sufficient and exact mathematical insight into this phenomenon based on the analysis and numerical solution of the quasi-one-dimensional and three-dimensional compressible Euler equations. In the former case, we show the non-uniqueness of solution analytically. Further we mention some useful properties of sonic points which are a potential source of the non-uniqueness. Finally, we illustrate the presence of the nonuniqueness also in the three-dimensional model, using an axisymmetric finite volume scheme. Both analytical and numerical examples are presented.

MSC 2000. 35L65, 35L67, 76N10, 76N15, 76R10

Keywords. non-uniqueness, inviscid gas flow, compressible Euler equations, quasi-one-dimensional, axisymmetric three-dimensional, finite volume method

⋆ This research was in part supported by the Grant No. 201/01/1200 of the Grant Agency of the Czech Republic.

This is the preliminary version of the paper.

1 Introduction

The question of uniqueness of the exact as well as numerical solution is always very important if a nonlinear problem is solved. The theory of almost unidirectional inviscid flow of perfect gases in tubes, ducts, pipes and nozzles is undoubtedly a relatively well-explored discipline (see e.g. [3,4,5,7,6,9,12,14,15]), which deserves a permanent industrial attention. Nevertheless, only a surprisingly few remarks can be found in the literature on the non-unique behavior that these flows can exhibit at sonic and transonic regimes in nontrivial axisymmetric geometries. It is our aim to provide some analytical and numerical insight into this phenomenon in the present paper, which is based on the article [10].

As compared with [10], we are more brief concerning the derivation of the model, the recursive algorithm for the construction of all exact solutions to our problem and the axi-symmetric finite volume method. On the other hand, we explain in more detail the analytical example where non-uniqueness of the stationary quasi-one-dimensional compressible Euler equations is rooted because this is a source of non-unique behaviour of almost-unidirectional flow of gases, which can be observed also in more dimensions.

2 Quasi-one-dimensional model

In this section we will analyze a basic model of almost unidirectional inviscid compressible flow, which at the same time offers a sufficiently complex description of the flow and is sufficiently transparent to have an analytical solution. Obviously, for real-life industrial simulations, advanced quasi-one-dimensional models (such as, e.g., the Fanno model discussed in [4,5,6,9] including wall drag and turbulence effects are recommended.

Let us consider a bounded interval $I = [x_a, x_b] \subset (-\infty, \infty)$, real constants $0 < R_{min} \le R_{max}$ and a bounded real function $r(x) : I \to [R_{min}, R_{max}]$ describing the radius of an axisymmetric pipe or nozzle. For simplicity, we suppose that $r(x)$ is once continuously differentiable in (x_a, x_b) and continuous in I, but the results presented in this paper are valid also for $r(x)$ continuous and only piecewise smooth. We define a varying cross-section $a(x) = \pi r^2(x)$ for $x \in I$.

We consider the standard stationary quasi-one-dimensional compressible Euler equations for perfect gases (see, e.g., [3,4,7,10,12,14,15]). Due to their hyperbolicity and the assumed smoothness of the radius r, discontinuous but piecewise smooth solutions can be expected. Along the smooth parts of the solutions, the partial differential equations can be rewritten into the following system of three non-linear algebraic equations

$$a(x)\varrho(x)u(x) = m, \tag{1}$$

$$\frac{\kappa p(x)}{(\kappa - 1)\varrho(x)} + \frac{1}{2}u^2(x) = h, \tag{2}$$

$$\frac{p(x)}{\varrho^\kappa(x)} = s \tag{3}$$

expressing the conservation of mass m, enthalpy h and entropy S (where $S = c_v \ln s + \text{const.}$, c_v being the specific heat of the gas), respectively. Here $\varrho(x)$, $u(x)$, $p(x)$, $e(x)$ mean the density, velocity, pressure and total energy density, respectively, and $\kappa \in (1,3)$ is a real constant. Derivation of (1) and (2) is straightforward, and a detailed derivation of (3) and further properties of the entropy S can be found, e.g., in [1,3,4,5,7,9,15].

In the quasi-one-dimensional case, all discontinuities are necessarily *shocks* (simultaneous discontinuities in all ϱ, u and p) as contact discontinuities are not relevant (see, e.g., [15]). Rankine-Hugoniot conditions, which can be derived from the weak formulation of the compressible Euler equations, imply that m, h are conserved also at shocks (see, e.g., [15]). It is known that the entropy S is not conserved at shocks where it rises discontinuously obeying the Rankine-Hugoniot relation

$$p(x_+) = p(x_-)\frac{2\kappa M(x_-)^2 - \kappa + 1}{\kappa + 1}. \tag{4}$$

Here $x \in I$ and $p(x_+)$, $p(x_-)$, $M(x_-)$ mean the downstream pressure limit and upstream pressure and Mach number limits at the shock, respectively.

For the sake of completeness, let us recall the speed of sound $c(x)$ and the Mach number $M(x)$ defined by

$$c(x) = \sqrt{\kappa p(x)/\varrho(x)},\ M(x) = |u(x)|/c(x), \tag{5}$$

respectively. Flow is called *subsonic* where $M(x) < 1$, *sonic* where $M(x) = 1$ and *isentropic* where the quantity s from (3) is conserved.

Lemma 1. *In inviscid compressible flow, shocks cannot occur in subsonic or sonic flow regions. Flow leaves a shock always at subsonic regime. The entropy S and the quantity s defined in (3) are discontinuous at shocks and always increasing with respect to the flow direction. Both of these quantities stay conserved except for shocks.*

Proof. See any basic book on fluid mechanics, e.g. [1].

Almost always we are interested in a subsonic inlet. These considerations lead us to a mathematically exact formulation of the problem of our interest:

Problem 2. Let I, r, a be as described above. Consider boundary data $\varrho_a > 0$, $u_a > 0$, $p_a > 0$ such that $M_a = u_a/\sqrt{\kappa p_a/\varrho_a} \le 1$, and p_b such that $p_a \ge p_b > 0$. Let us put $m = a(x_a)\varrho_a u_a$, $h = \kappa p_a/((\kappa - 1)\varrho_a) + u_a^2/2$ according to (1), (2), respectively. For a finite set $\mathbb{D} \subset (x_a, x_b)$ (corresponding to shocks), partitioning (x_a, x_b) into a finite number of non-overlapping open intervals $I_1, I_2, \ldots, I_d$ (ordered from the left to the right), we consider a sequence of constants $p_a/\varrho_a^\kappa = s_1 < s_2 < \ldots < s_d$. The set $\mathbb{D}$ and real functions ϱ, u, p defined in $I \setminus \mathbb{D}$ are sought such that

1. ϱ, u, p are bounded, positive and smooth in $I \setminus \mathbb{D}$;
2. ϱ, u, p satisfy (1), (2) in $I \setminus \mathbb{D}$ with the constants m, h, respectively;
3. ϱ, p satisfy (3) in $I \setminus \mathbb{D}$ with $p/\varrho^\kappa = s_k$ in I_k for all $1 \le k \le d$;
4. ϱ, u, p satisfy (4) at all $x \in \mathbb{D}$;
5. $\varrho(x_a) = \varrho_a$, $u(x_a) = u_a$, $p(x_a) = p_a$, $p(x_b) = p_b$.

Remark 3. If Problem 2 has a solution, the value of u_a is determined (except for a few degenerate situations) by ϱ_a, p_a and p_b as described, e.g., in [3,12]. Despite the fact that the prescription of u_a seems to be an unnecessary additional restricting condition, it allows us not to deal with its computation here and improves the clarity of further considerations.

In the next section, we are going to present a simple problem where the non-uniqueness of solution can be shown analytically in a constructive way.

3 Example of a problem with a non-unique solution

Lemma 4. *If Problem 2 has a solution ϱ, u, p, there is a unique pair of values $\varrho_b, u_b > 0$ such that $\varrho(x_b) = \varrho_b$, $u(x_b) = u_b$. If $p_b = p_a$ it is $\varrho_b = \varrho_a$, $u_b = u_a$.*

Proof. Let us assume that a solution to Problem 2 exists. Equations (1), (2), expressing the conservation of m, h in $I \setminus \mathbb{D}$, yield a quadratic equation for $u(x_b) = u_b$. It is easy to see that this equation has always one positive and one negative root. The negative one is meaningless with respect to (1). Relation (1) yields also the density ϱ_b. The rest is easy to see.

Theorem 5. *Let the radius r be as described in Section 2 and moreover satisfy $r(x_a) = r(x_b) = r_0 > 0$, $r(x) > r_0$ for all $x \in (x_a, x_b)$. Let the boundary data of Problem 2 satisfy $M_a = u_a/\sqrt{\kappa p_a/\varrho_a} = 1$, $p_b = p_a$. Then, Problem 2 has exactly two solutions in I, both of them smooth in (x_a, x_b).*

Proof. According to Lemma 4, a solution of Problem 2 must satisfy $\varrho(x_b) = \varrho_a$. As $p_a/\varrho_a^\kappa = p(x_b)/\varrho^\kappa(x_b)$, Lemma 1 yields that $\mathbb{D} = \emptyset$. Thus, relation (4) is not relevant. Putting (1) and (3) into (2), we obtain

$$\frac{\kappa s}{\kappa - 1}\varrho^{\kappa-1}(x) + \frac{m^2}{2a^2(x)\varrho^2(x)} - h = 0 \tag{6}$$

for all $x \in I$. For an $x \in I$, equation (6) can be written in the form

$$f_\rho(\varrho(x)) = 0, \tag{7}$$

with the implicit function

$$f_\rho(\xi) = \frac{\kappa s}{\kappa - 1}\xi^{\kappa-1} + \frac{m^2}{2a^2(x)\xi^2} - h \tag{8}$$

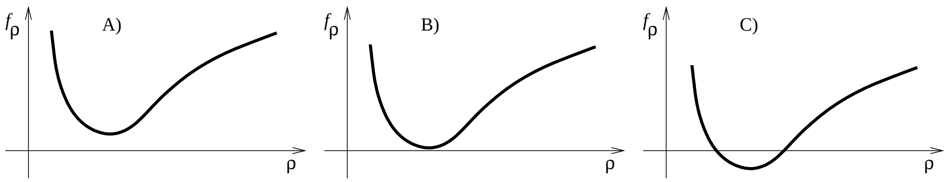

Fig. 1. Function f_ρ; situations A) with no real root, B) exactly one real root, C) two different real roots.

defined for all $\xi \in (0, \infty)$. The function f_ρ depicted in the Fig. 1 is smooth and achieves its only minimum at $\xi_{min} = \left(m^2/(\kappa a^2 s)\right)^{\frac{1}{\kappa+1}}$. The derivative f_ρ' is negative in $(0, \xi_{min})$ and positive in (ξ_{min}, ∞). It is $f_\rho(\xi_{min}) = 0$ both for x_a and x_b. For all $x \in I$, the value of $f_\rho(\xi_{min})$ is a decreasing function of the cross-section $a(x)$. Thus, the equation (7) has exactly one positive root $\varrho_1(x) = \varrho_2(x) = \varrho_a$ for $x = x_a$, exactly two positive roots $0 < \varrho_1(x) < \varrho_2(x)$ for all $x \in (x_a, x_b)$ and exactly one positive root $\varrho_1(x) = \varrho_2(x) = \varrho_a$ for $x = x_b$. The implicit function theorem immediately yields that the functions $\varrho_1(x), \varrho_2(x)$ are smooth in (x_a, x_b). Putting the solutions $\varrho_1(x), \varrho_2(x)$ into the equation (1), we obtain two different positive smooth solutions $u_1(x), u_2(x)$ for the velocity. Finally, using (3), we obtain the corresponding solutions $p_1(x), p_2(x)$ for pressure.

3.1 Properties of sonic points

As we have seen in the previous paragraph, *sonic points* (points $x \in I$ such that $M(x) = 1$) play an important role in the existence of non-unique solutions to Problem 2. It is our aim to mention some of their further useful properties in this paragraph.

Lemma 6. *Let $\mathcal{D}, \varrho, u, p$ solve Problem 2. Let $x \in I \setminus \mathcal{D}$ such that $M(x) = 1$. Then the solution of Problem 2 at x has the unique form*

$$u(x) = \left(\frac{2h(\kappa - 1)}{\kappa + 1}\right)^{1/2}, \;\; \varrho(x) = \frac{m}{a(x)u(x)}, \;\; p(x) = \frac{(\kappa - 1)\varrho(x)\left[h - u^2(x)/2\right]}{\kappa}. \tag{9}$$

Proof. Immediately from (1), (2) using (5).

Lemma 7. *Let $\mathcal{D}, \varrho, u, p$ solve Problem 2. Let $x_1, x_2 \in I \setminus \mathcal{D}$, $x_1 < x_2$, $M(x_1) = M(x_2) = 1$ and $a(x_1) = a(x_2)$. Then ϱ, u, p are continuous in $[x_1, x_2]$.*

Proof. Using Lemma 6 with $a(x_1) = a(x_2)$, we obtain $\varrho(x_1) = \varrho(x_2)$, $p(x_1) = p(x_2)$. Thus, $p(x_1)/\varrho^\kappa(x_1) = p(x_2)/\varrho^\kappa(x_2)$. Lemma 1 implies the continuity of ϱ, u, p in $[x_1, x_2]$. This obviously means that there is no $\tilde{x} \in \mathcal{D}$, $x_1 < \tilde{x} < x_2$.

Lemma 8. *Let $\mathcal{D}, \varrho, u, p$ solve Problem 2. Let $x_1, x_2 \in I \setminus \mathcal{D}$, $x_1 < x_2$, $M(x_1) = M(x_2) = 1$ and $a(x_1) \neq a(x_2)$. Then none of ϱ, u, p can be continuous in $[x_1, x_2]$. Moreover, necessarily it is $a(x_1) < a(x_2)$.*

Proof. Lemma 6 with $a(x_1) \neq a(x_2)$ yields that $\varrho(x_1) \neq \varrho(x_2)$, $p(x_1) \neq p(x_2)$. This and conservation of m, h in $I \setminus \mathcal{D}$ yield that $p(x_1)/\varrho^\kappa(x_1) \neq p(x_2)/\varrho^\kappa(x_2)$. Lemma 1 implies that $p(x_1)/\varrho^\kappa(x_1) < p(x_2)/\varrho^\kappa(x_2)$. Relation (9) yields that this is only possible if $a(x_1) < a(x_2)$.

Lemma 9. *Let $\mathcal{D}, \varrho, u, p$ solve Problem 2. Let $x_1, x_2 \in I \setminus \mathcal{D}$, $x_1 < x_2$, and let ϱ, u, p be continuous in $[x_1, x_2]$.*
a) If $M(x_1) < 1$ and $r(x)$ is decreasing in $[x_1, x_2]$ then $M(x)$ is increasing in $[x_1, x_2]$, but the relation $M(x) < 1$ is preserved in $[x_1, x_2)$.
b) If $M(x_1) < 1$ and $r(x)$ is increasing in $[x_1, x_2]$ then $M(x)$ is decreasing in $[x_1, x_2]$, and obviously $M(x) < 1$ in $[x_1, x_2]$.
c) If $M(x_1) > 1$ and $r(x)$ is decreasing in $[x_1, x_2]$ then $M(x)$ is decreasing in $[x_1, x_2]$, but the relation $M(x) > 1$ is preserved in $[x_1, x_2)$.
d) If $M(x_1) > 1$ and $r(x)$ is increasing in $[x_1, x_2]$ then $M(x)$ is increasing in $[x_1, x_2]$, and obviously $M(x) > 1$ in $[x_1, x_2]$.

Proof. We put $s_1 = p(x_1)/\varrho^\kappa(x_1)$. Let $x \in (x_1, x_2)$. For ϱ, u, p continuous, relations (1), (2) and (3) with (5) yield

$$u^2(x) = \frac{2(\kappa - 1)h}{2/M^2(x) + \kappa - 1}, \tag{10}$$

$$\varrho(x) = \frac{m}{a(x)u(x)}, \tag{11}$$

$$s_1 = \frac{a^{\kappa-1}(x)(2(\kappa-1)h)^{\frac{\kappa+1}{2}}}{\kappa m^{\kappa-1} M^2(x) \left[2/M^2(x) + \kappa - 1\right]^{\frac{\kappa+1}{2}}}. \tag{12}$$

Relation (12) can be written as

$$M^2(x) \left(2/M^2(x) + \kappa - 1\right)^{\frac{\kappa+1}{2}} \frac{1}{a^{\kappa-1}(x)} = \frac{(2(\kappa-1)h)^{\frac{\kappa+1}{2}}}{\kappa m^{\kappa-1} s_1} = \text{const.} \tag{13}$$

We consider (13) as an implicit function for the Mach number M. Analysis of the shape of its solution (with respect to the monotone of r supposed in a) to d)) yields the monotone behavior of M. This analysis also yields that in a) and c), the existence of an $x \in (x_1, x_2)$ such that $M(x) = 1$ is contradictory to (13).

Corollary 10. *Without loss of generality, we can assume that $M(x_a) \neq 1$ in the case that the radius r does not have a local minimum at x_a (namely, using the implicit function f_ϱ from the proof of Theorem 5, it can be shown that there would be no solution to Problem 2). Thus, Lemma 9 excludes all possibilities for the existence of sonic points except for such $x \in I$ where the radius r achieves a local minimum.*

Corollary 10 together with Lemma 6 will play an important role in the construction of multiple solutions as we will be able to evaluate solution of Problem 2 at these points in the pipe or nozzle a priori.

4 Example: a multiple nozzle

In this section we deal with non-unique solutions to Problem 2 corresponding to a nontrivial function r. We present analytical results obtained with a recursive algorithm described in [10] as well as numerical results obtained by a suitable axisymmetric finite volume scheme for three-dimensional compressible Euler equations derived in [10]. The function

$$r(x) = \begin{cases} -\frac{\cos(10\pi(x-0.05))}{50} + 0.0265, & x \in [x_a, 0.05], \\ -\frac{\sin(10\pi x)}{250} + x/100 + 1/100, & x \in [0.05, x_b] \end{cases} \tag{14}$$

is considered in the interval I given by $x_a = -0.05$ m and $x_b = 0.75$ m as shown in Fig. 2.

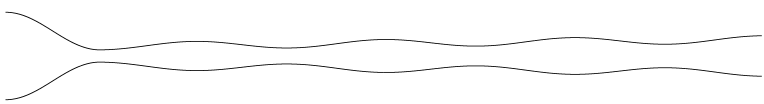

Fig. 2. Geometry of the multiple nozzle.

Boundary conditions are chosen as $p_a = 60000$ Pa, $\theta_a = 368.16$ K (where θ_a is the inlet temperature) and $p_b = 13000$ Pa. Density ϱ_a is evaluated directly from the perfect gas state equation. For this set of boundary data, the nozzle works in the *Laval regime* with $M(0.05) = 1$ and the value of the subsonic inlet velocity is computed as $u_a = 5.42$ m/s.

The interval I is divided equidistantly into $N_{elem} = 2000$ finite volumes. In Figures 3 to 6, we show four different analytical solutions to Problem 2. In Figures 7 to 10, steady state results of the corresponding axisymmetric computation are shown. Let us remark that we need one more boundary condition at the subsonic inlet in the axisymmetric case in comparison with the quasi-one-dimensional one. The reason is that the number of incoming characteristics in the axisymmetric case is greater. To be compatible with the quasi-one-dimensional case, we prescribe in addition to an inlet density ϱ_a and pressure p_a also zero radial component of the subsonic inlet velocity $u_r = 0$. In agreement with the quasi-one-dimensional case, pressure p_b is prescribed at the subsonic outlet. At last, we compare the analytical results with the numerical ones (corresponding to axial cutlines of the axisymmetric geometry) in Figures 11 to 14. Here analytical solutions are depicted by solid lines and the finite volume cutlines by dashed ones. A structured triangular grid with 12 000 finite volumes was used for the axisymmetric finite volume computation.

4.1 Analytical quasi-1D solutions

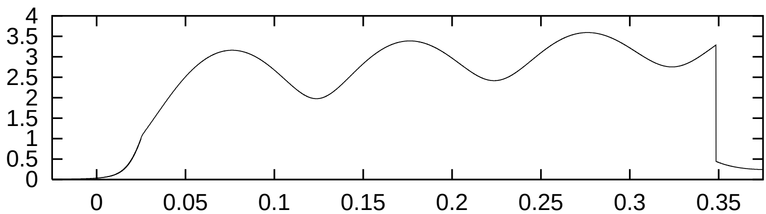

Fig. 3. Mach number, analytical solution with one shock.

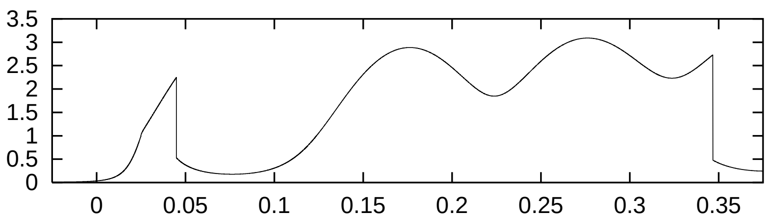

Fig. 4. Mach number, analytical solution with two shocks.

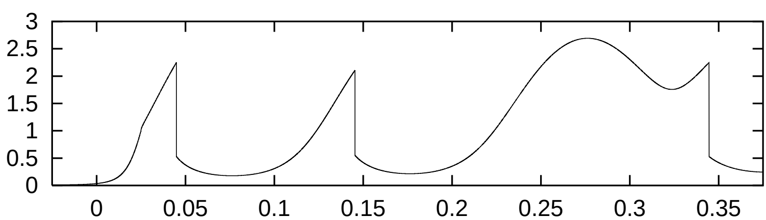

Fig. 5. Mach number, analytical solution with three shocks.

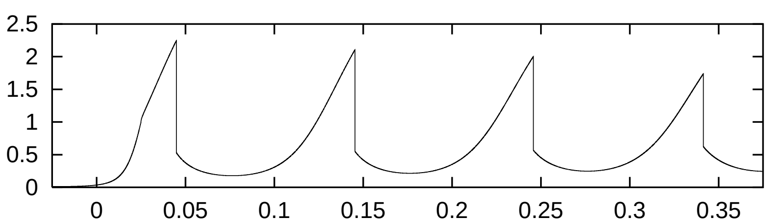

Fig. 6. Mach number, analytical solution with four shocks.

4.2 Axi-symmetric finite volume solutions

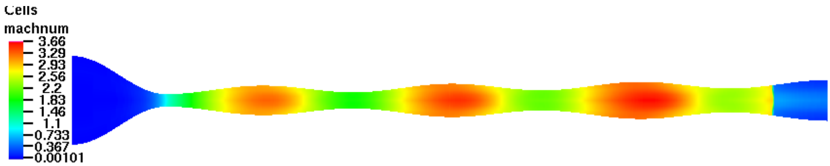

Fig. 7. Mach number color map, FV solution with one shock.

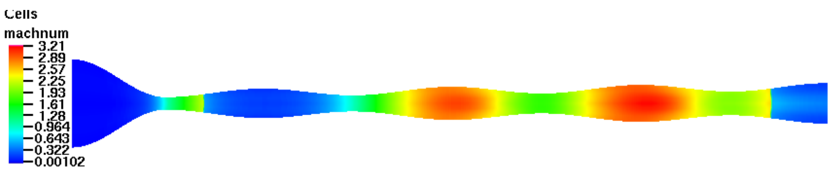

Fig. 8. Mach number color map, FV solution with two shocks.

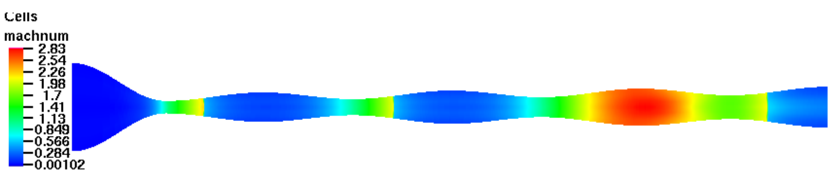

Fig. 9. Mach number color map, FV solution with three shocks.

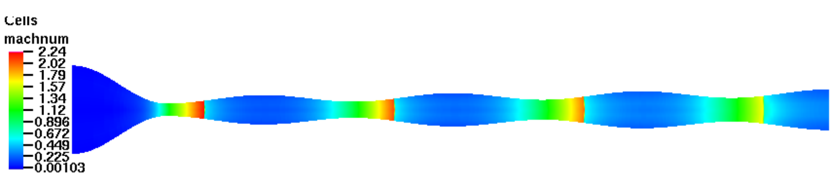

Fig. 10. Mach number color map, FV solution with four shocks.

4.3 Comparison of quasi-1D and axi-symmetric solutions

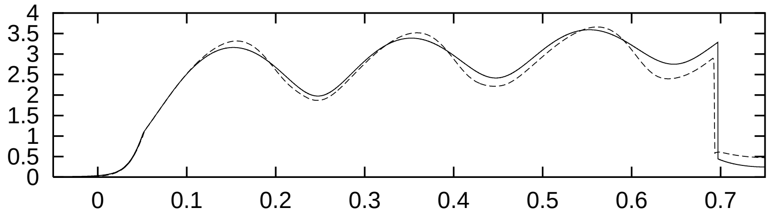

Fig. 11. Mach number, analytical and FV solutions with one shock.

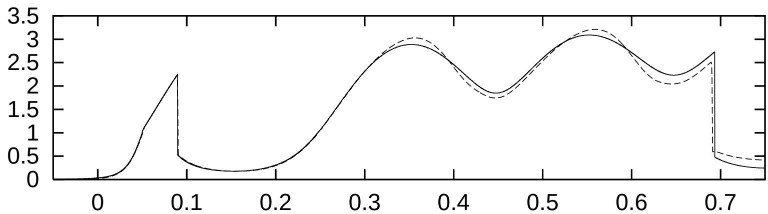

Fig. 12. Mach number, analytical and FV solutions with two shocks.

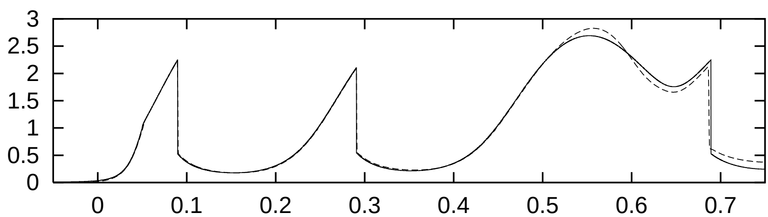

Fig. 13. Mach number, analytical and FV solutions with three shocks.

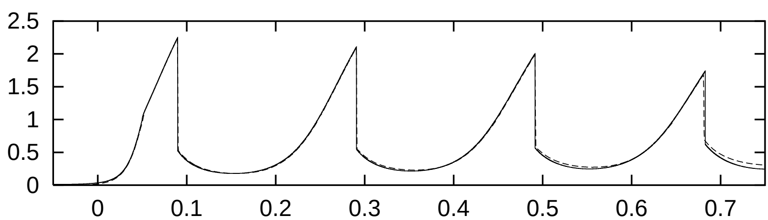

Fig. 14. Mach number, analytical and FV solutions with four shocks.

References

1. M. Feistauer, Mathematical Methods in Fluid Dynamics (Longman Scientific & Technical, Harlow, 1993).
2. J. Felcman and P. Šolín, Construction of the Osher-Solomon scheme for 3D Euler equations, East-West J. Numer. Math. 6 (1998) 43–64.
3. C. Hirsch, Numerical Computation of Internal and External Flows, Vol. 2 (J. Wiley and Sons, 1990).
4. D. D. Knight, Inviscid Compressible Flow, in: The Handbook of Fluid Dynamics (CRC, 1998).
5. L. D. Landau, E. M. Lifschitz, Fluid Mechanics (Pergamon Press, 1959).
6. H. Ockendon, J. R. Ockendon, The Fanno model for turbulent compressible flow, Preprint, OCIAM, Mathematical Institute, Oxford University, 2001.
7. H. Ockendon, A. B. Taylor, Inviscid Fluid Flow (Springer-Verlag, 1983).
8. S. Osher, F. Solomon, Upwind difference schemes for hyperbolic systems of conservation laws, Math. Comp. 38 (1982) 339–374.
9. A. H. Shapiro, The Dynamics and Thermodynamics of Compressible Fluid Flow, Vol. 1 (The Ronald Press Co., New York, 1953).
10. P. Šolín, K. Segeth, Non-uniqueness of almost unidirectional inviscid compressible flow, submitted to J. Comput. Phys..
11. J. L. Steger, R. F. Warming, Flux vector splitting of the inviscid gas dynamics equations with applications to finite difference methods, J. Comput. Phys. 40 (1981) 263–293.
12. E. Truckenbrodt, Fluidmechanik, Band 2 (Springer-Verlag, Berlin-Heidelberg-New York, 1980).
13. G. Vijayasundaram, Transonic flow simulation using upstream centered scheme of Godunov type in finite elements, J. Comput. Phys. 63 (1986) 416 – 433.
14. A. J. Ward-Smith, Internal Fluid Flow, in: The Fluid Dynamics of Flow in Pipes and Ducts (Clarendon Press, Oxford, 1980).
15. P. Wesseling, Principles of Computational Fluid Dynamics (Springer-Verlag, 2000).