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Quasilinear Elliptic Dirichlet Problem in Nonregular Domains*

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Abstract. We present the solvability result for the Dirichlet problem to nondivergent quasilinear elliptic equations of the second order in weighted Kondrat'ev spaces in the case when the boundary of a domain may include singularities — conical points or arbitrary codimensional edges.

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We consider the boundary value problem

$$-a^{ij}(x,u,Du)D_iD_ju + a(x,u,Du) = 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

where Ω is a domain in $\mathbb{R}^n$, $(n \geqslant 2)$, with compact closure $\overline{\Omega}$ and with nonregular boundary $\partial\Omega$.

The term "nonregular" means that $\partial\Omega$ contains a $(n-m)$-dimensional submanifold $\mathcal{M}$ (an "edge" for $m < n$ or a conical point for $m = n$), satisfying the following condition: for all $x^0 \in \mathcal{M}$ there exist a neighborhood $U(x^0) \subset \mathbb{R}^n$ and a diffeomorphism $\Psi_{(x^0)} : U(x^0) \to \mathbb{R}^n$, such that

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A note on results published in [1].

(i) $\Psi_{(x^0)}(U(x^0)\cap\Omega)=\{x\in\mathcal{K}_m(G):|x'|<\rho_0,|x''|<\rho_0\}$.
Here $\mathcal{K}_m(G)=\mathbb{K}_m(G)\times\mathbb{R}^{n-m}$, $\mathbb{K}_m(G)$ stands for an open m-dimensional cone cutting on the unit sphere S^m a domain G with smooth boundary, $x=(x',x'')$, $x'\in\mathbb{R}^m$, $x''\in\mathbb{R}^{n-m}$ and $|x'|$, $|x''|$ denote corresponding Euclidean norms. Note also that G depends on x^0 while $\rho_0\leqslant 1$ does not depend.

(ii) $\Psi_{(x^0)}(U(x^0)\cap\partial\Omega)=\{x\in\partial\mathcal{K}_m(G):|x'|<\rho_0,|x''|<\rho_0\}$,

(iii) $\Psi_{(x^0)}(x^0)=0, \qquad \Psi'_{(x^0)}(x^0)=I_n$,

(iv) the norms of Jacobians $\Psi'_{(x^0)}(x)$ and $(\Psi^{-1}_{(x^0)})'(\Psi_{(x^0)}(x))$ are bounded uniformly with respect to $x^0\in\mathcal{M}$ and $x\in U(x^0)$,

(v) $\mathcal{K}_m(G)\subset\{x\in\mathbb{R}^m:\widehat{x',x_1}<\theta<\frac{\pi}{2}\}$ for all $x^0\in\mathcal{M}$, and θ does not depend on x^0.

Setting $d(x)=\mathrm{dist}\{x,\mathcal{M}\}$ we introduce the scale of weighted spaces $\mathbb{L}_{r,(\alpha)}(\Omega)$ with the norm

$$\|u\|_{r,(\alpha),\Omega}=\|u\cdot(d(x))^\alpha\|_{L_r(\Omega)},$$

and the scale of Kondrat'ev spaces $\mathbb{V}^2_{r,(\alpha)}(\Omega)$ with the norm

$$\|u\|_{\mathbb{V}^2_{r,(\alpha)}(\Omega)}=\|D(Du)\|_{r,(\alpha),\Omega}+\|Du\cdot(d(x))^{-1}\|_{r,(\alpha),\Omega}+\|u\cdot(d(x))^{-2}\|_{r,(\alpha),\Omega}.$$

Finally, the notation $\partial\Omega\in\mathbb{V}^2_{r,(\alpha)}$ with $\alpha<1-n/r$ is understood as follows:

1) $\partial\Omega\setminus\mathcal{M}\in W^2_{r,loc}$;
2) for all points $x^0\in\mathcal{M}$ the matrix $D^2\Psi_{(x^0)}$ belongs to $\mathbb{L}_{r,(\alpha)}(U(x^0))$. Moreover the norms $\|D^2\Psi_{(x^0)}\|_{r,(\alpha)}$ are bounded uniformly with respect to $x^0\in\mathcal{M}$.

Assume that (a^{ij}) in (1) is a symmetric matrix and the following natural structure conditions hold for all $x\in\Omega$, $z\in\mathbb{R}^1$, $p\in\mathbb{R}^n$:

$$\nu|\xi|^2\leqslant a^{ij}(x,z,p)\xi_i\xi_j\leqslant\nu^{-1}|\xi|^2,\quad \forall\xi\in\mathbb{R}^n,\quad \nu=\mathrm{const}>0, \tag{A0}$$

$$|a(x,z,p)|\leqslant\mu|p|^2+b(x)|p|+\Phi_1(x),\qquad \mu=\mathrm{const}>0, \tag{A1}$$

$$b,\Phi_1\in\mathbb{L}_{r,(\alpha)}(\Omega),\quad \alpha<1-n/r,\quad n<r<\infty; \tag{2}$$

$$\left|\frac{\partial a^{ij}(x,z,p)}{\partial p_k}\right|\leqslant\frac{\mu}{1+|p|},\quad \left|\frac{\partial a^{ij}(x,z,p)}{\partial z}p_k+\frac{\partial a^{ij}(x,z,p)}{\partial x_k}\right|\leqslant\mu|p|+\Phi_2(x), \tag{A2}$$

$$\Phi_2\in\mathbb{L}_{q,(\alpha_1)}(\Omega),\quad \alpha_1<1-n/q,\quad n<q<\infty. \tag{3}$$

Before stating the main result we need to introduce some notations. Let $\widehat{\theta}(\theta,\nu)$ be the solution of the equation

$$\mathrm{ctg}(\widehat{\theta})=\nu\cdot\mathrm{ctg}(\theta),\qquad \widehat{\theta}\in\Big]0,\frac{\pi}{2}\Big[.$$

Let also $\widehat{\Lambda}(m,\theta)$ be the first eigenvalue of the Dirichlet problem for the Laplace-Bel'trami operator on the spherical "cap" $\{x\in\mathbb{R}^m:\widehat{x',x_1}<\theta\}\cap S^m$, while $\widehat{\omega}$ be a positive solution of the equation $\omega^2+(m-2)\omega-\widehat{\Lambda}=0$.

Theorem 1 (Solvability in weighted spaces). *Let the following conditions be fulfilled:*

(a) $r > \max\{n, \frac{n-m}{\widehat{\omega}-1}\}, \quad \alpha \in]2 - \frac{m}{r} - \widehat{\omega}, 1 - \frac{n}{r}[, \quad \partial\Omega \in \mathbb{V}^2_{r,(\alpha)}$,

(b) *for all solutions $u^{[\tau]}(\cdot) \in \mathbb{V}^2_{r,(\alpha)}(\Omega)$, $\tau \in [0,1]$, of the family of problems:*

$$\begin{gathered} \tau(-a^{ij}(x,u,Du)D_iD_ju + a(x,u,Du)) - (1-\tau)\Delta u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 \end{gathered} \tag{4}$$

the estimate $\|u^{[\tau]}(\cdot)\|_\Omega \leqslant M_0$ holds true,

(c) *the conditions (A0)—(A2), (2)—(3) are fulfilled for $|z| \leqslant M_0$,*

(d) *the function $a(\cdot, z, p)$ regarded as an element of the space $\mathbb{L}_{r,(\alpha)}(\Omega)$ is continuous with respect to (z,p).*

Then for all $\tau \in [0,1]$ the problem (4) has a solution $\widehat{u}^{[\tau]}(\cdot) \in \mathbb{V}^2_{r,(\alpha)}(\Omega)$. In particular, $\widehat{u}^{[1]}(\cdot)$ is a solution of the problem (1).

For details and proof we refer the reader to [1].

References

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