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Distributed Delayed Competing Predators

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Abstract. We propose a model of competing species where the dynamic of the predators depend on the past history of the prey by mean of distributed delay that takes an average of the Michaelis -Menten functional response of the prey population. We show that the system is pointwise dissipative and the existence of a global attractor of the solutions of this model.

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1 Statement of the model

In this paper we improve the model of two competing predators species proposed by S. B. Hsu, S. P. Hubbell and P. Waltman in [6] and [5]. There, the authors have considered a renewable resourse with reproductive properties, a more classic prey, and the predators compete purely exploitatively for the prey without interference between rivals. Our purpose here, in a more realistic fashion, is to introduce a distributed delay in the equations in the same way that in [2] and [9].

Thus we start with the model

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$$
\begin{aligned}
S'(t) &= S(t)\left(1-\frac{S(t)}{K}\right) - \frac{m_1 x_1(t) S(t)}{a_1 + S(t)} - \frac{m_2 x_2(t) S(t)}{a_2 + S(t)} \\
x_1'(t) &= \frac{m_1 x_1(t) S(t)}{a_1 + S(t)} - D_1 x_1(t) \\
x_2'(t) &= \frac{m_2 x_2(t) S(t)}{a_2 + S(t)} - D_2 x_2(t) .
\end{aligned}
$$

that have been analyzed in [6]. In the equations $x_1(t)$, $x_2(t)$ denote the predator populations that compete by the prey specie, $S(t)$ at the time t. Both species have access to the prey and compete only by lowering the population of shared prey. For death rates it is assumed that the number dying is proportional to the number currently alive. In the absence of predators the prey grows logistically and the predator's functional response obey the Michaelis-Menten kinetic too so called the Holling type curve. The parameter m_i is the maximum growth (birth) rate of the ith predator, D_i is the death rate for the ith predator, a_i is the half-saturation constant for the ith predator, which is the prey density at which the functional response of the predator is half maximal. The parameter K is the carrying capacity for the prey population.

We shall consider the following modification of the predator-prey model considering that there are significative lags in the system by including a distributed delay to describe the time lag involved in the process of conversion of prey consumed into predators. More precisely,we are assuming in a more realistic fashion that the present level of the predators affects instantaneously the growth of the prey, but the growth of the predator is influenced by the amount of prey in the past. We are supposing that the predator grow up depending on the weight average time of the function of Michaelis-Menten of S over the past per predator. Thus, the model takes the form

$$
\begin{aligned}
S'(t) &= S(t)\left(1-\frac{S(t)}{K}\right) - \frac{m_1 x_1(t) S(t)}{a_1 + S(t)} - \frac{m_2 x_2(t) S(t)}{a_2 + S(t)} \\
x_1'(t) &= -D_1 x_1(t) + \int_{-\infty}^{t} \alpha_1 x_1(\tau) \exp(-\alpha_1 (t-\tau)) \left(\frac{m_1 S(\tau)}{a_1 + S(\tau)}\right) d\tau \qquad (1) \\
x_2'(t) &= -D_2 x_2(t) + \int_{-\infty}^{t} \alpha_2 x_1(\tau) \exp(-\alpha_2 (t-\tau)) \left(\frac{m_2 S(\tau)}{a_2 + S(\tau)}\right) d\tau.
\end{aligned}
$$

In the literature the kernel,

$$K(u) = \alpha \exp(-\alpha u), \qquad (2)$$

is called the weak kernel and is frequently used in biological modeling and clearly implies that the influence of the past is fading away exponentially and the number

$\frac{1}{\alpha_i}$ might be interpreted as the measure of the influence of the past. So, to smaller α_i, longer is the interval in the past in which the values of S are taken into account (see [3], [7]). This kernel has the property that

$$\int_{-\infty}^{t} \alpha_i \exp(-\alpha_i(t-\tau))d\tau = \int_{0}^{\infty} \alpha_i \exp(-\alpha_i s)ds = 1. \tag{3}$$

We show that this system is pointwise dissipative and therefore that there exists an global attractor for the solutions of the model.

2 An equivalent system

The adequate space for the initial data of our problem and some related notations is as follow:

Let BC_+^3 denote the Banach space of the bounded and continuous function mapping from the interval $(-\infty, 0]$ to $\mathbb{R}_+^3$, the 3-dimensional vectors with positive coordinates . From the general theory of integral differential equations(see [1] and [8]), for any initial data $\phi = (\phi_0, \phi_1, \phi_2) \in BC_+^3$, there exists a unique solution $\pi(\phi; t) := (S(\phi; t), x_1(\phi; t), x_2(\phi; t))$ for all $t > 0$ and $\pi(\phi; 0) \mid_{(-\infty,0]} = \phi$. Throughout, we will use $(S(t), x_1(t), x_2(t))$ to denote the solution $\pi(\phi; t)$ with $\phi \in BC_+^3$, when no confusion arises. By a positive solution $\pi(\phi; t)$ or $(S(t), x_1(t), x_2(t))$ of system (1), we shall mean the solution has initial condition $\phi \in BC_+^3$ and each component of the solution is positive for all $t > 0$.

However, the integrodifferential system (1) can be associated to an system ordinary differential equations in the following way:

We introduce the two new unknown functions, y_i, defined by

$$y_i(t) = \int_{-\infty}^{t} \alpha_i x_i(\tau) \exp(-\alpha_i(t-\tau)) \left(\frac{m_i S(\tau)}{a_i + S(\tau)} \right) d\tau, \quad i = 1, 2. \tag{4}$$

and using the the form of the weak kernel (2) and the *linear trick chain technique* (see [7]) we get the new system of ordinary differential equations,

$$\begin{aligned}
S'(t) &= S(t)\left(1 - \frac{S(t)}{K}\right) - \frac{m_1 x_1(t) S(t)}{a_1 + S(t)} - \frac{m_2 x_2(t) S(t)}{a_2 + S(t)} \\
x_1'(t) &= -D_1 x_1(t) + y_1(t) \\
x_2'(t) &= -D_2 x_2(t) + y_2(t) \\
y_1'(t) &= \alpha_1 \left(\frac{m_1 x_1(t) S(t)}{a_1 + S(t)} - y_1(t) \right) \\
y_2'(t) &= \alpha_2 \left(\frac{m_2 x_2(t) S(t)}{a_2 + S(t)} - y_2(t) \right).
\end{aligned} \tag{5}$$

Thus, system (1) is "equivalent" to this system of five-dimensional ordinary differential equations. We understand the relationship between the systems (1) and (5) as follows : If $(S, x_1, x_2) : [0, \infty) \to \mathbb{R}^3_+$ is the solution of (1) corresponding to the continuous and bounded initial functions $(\tilde{S}, \tilde{x_1}, \tilde{x_2}) : (-\infty, 0] \to \mathbb{R}^3_+$, then $(S, x_1, x_2, y_1, y_2) : [0, \infty) \to \mathbb{R}^5_+$ is solution of (5) with the initial conditions $S(0) = \tilde{S}(0)$, $x_1(0) = \tilde{x}_{10}$, $x_2(0) = \tilde{x}_{20}$ and

$$y_i(0) = \int_{-\infty}^{0} \alpha_i x_i(\tau) \exp(-\alpha_i(t-\tau)) \left(\frac{m_i S(\tau)}{a_i + S(\tau)} \right) d\tau, \quad i = 1, 2.$$

Conversely, if (S, x_1, x_2, y_1, y_2) is any solution of (5), defined on the entire real line and bounded on $(-\infty, 0]$, then y_i, $i = 1, 2$, is given by (2) so (S, x_1, x_2) satisfies (4).

In most of the results of the paper we will consider the "equivalent system" (5).

3 Pointwise dissipativity and existence of an global attractor

The first lemma is basic for the well-posedness of the model (1). The result indicates that the model (1) possesses the property that positive initial data yield positive solutions.

Lemma 1 (Positivity). *For any $\phi \in BC^3_+$ with $\phi_0 > 0$, $\phi_i > 0$, $i = 1, 2$, the solution $\pi(\phi; t)$ remains positive for all $t > 0$.*

Proof. Clearly,

$$S(t) = S(0) \exp\left(\int_0^t \left\{ \left(1 - \frac{S(\tau)}{K}\right) - \frac{m_1 x_1(\tau)}{a_1 + S(\tau)} - \frac{m_2 x_2(\tau)}{a_2 + S(\tau)} \right\} d\tau \right),$$

and so $S(t) > 0$ for all $t > 0$ since $S(0) > 0$.

To show that $x_i(\phi; t) > 0$ for all $t > 0$, we suppose the contrary, that it is not true. Let

$$\bar{t} = inf\,\{t > 0 : x_i(\phi; t) = 0 \;\; and \;\; x_i(\theta) > 0, \; 0 \le \theta < t\} < \infty.$$

Then $x_i(\bar{t}) = 0$ and $x_i'(\bar{t}) \le 0$ and.But from (1), we have

$$\begin{aligned} x_i'(\bar{t}) &= -D_i x_i\,(\bar{t}) + \int_{-\infty}^{\bar{t}} \alpha_i x_i(\tau) \exp(-\alpha_i(\bar{t}-\tau)) \left(\frac{m_i S(\tau)}{a_i + S(\tau)} \right) d\tau \\ &= \int_{-\infty}^{\bar{t}} \alpha_i x_i(\tau) \exp(-\alpha_i(\bar{t}-\tau)) \left(\frac{m_i S(\tau)}{a_i + S(\tau)} \right) d\tau > 0. \end{aligned}$$

This is a contradiction. Therefore, $x_i(\phi; t) > 0$ for any $t > 0$.

This lemma implies that the set

$$\mathbb{E} = \{(S, x_1, x_2, y_1, y_2) \in \mathbb{R}^5 \mid S > 0, x_1 > 0, x_2 > 0, y_1 > 0, y_2 > 0\}$$

is invariant under the flow induced by the system (5).

The following theorem has to do with the property of pointwise dissipativity.

Theorem 2 (Pointwise Dissipativity). *All positive solutions of model (1) are bounded for $t > 0$. Moreover, system (5) is pointwise dissipative and the absorbing set (into which every solution eventually enters and remains) is given by*

$$B = [0, K] \times [0, L(D_1)] \times [0, L(D_2)] \times [0, \frac{K}{p} + 1] \times [0, \frac{K}{p} + 1], \tag{6}$$

where $L(D) = \frac{K}{pD} + \frac{1}{D} + 1$ *and* $p = \min\{1, \alpha_1, \alpha_2\}$.

Proof. Of the equation for S in (5) it is easy to see that we can use an similar argument as in the proof of Lemma 3.1 in [6] to obtain the boundedness of $S(t)$. Precisely, for sufficiently small $\varepsilon > 0$ there exists T depending only on $S(0)$, such that $S(t) < K + \varepsilon$, for $t > T$.

Now let

$$W = S + \frac{y_1}{\alpha_1} + \frac{y_2}{\alpha_2}$$

then, for $t > T$ we have,

$$\begin{aligned}
W' &= S' + \frac{y_1'}{\alpha_1} + \frac{y_2'}{\alpha_2} \\
&= S(1 - \frac{S}{K}) - y_1 - y_2, \\
&< K + \varepsilon - S - \alpha_1(\frac{y_1}{\alpha_1}) - \alpha_2(\frac{y_2}{\alpha_2}), \\
&< K + \varepsilon - p\{S - \frac{y_1}{\alpha_1} - \frac{y_2}{\alpha_2}\},
\end{aligned}$$

where $p = \min\{1, \alpha_1, \alpha_2\}$. And so,

$$W' < K - pW.$$

This clearly implies the uniform boundedness of y_1 and y_2. Note too that taking into account the equations for x_i in the equations (5), the boundedness of $y_i(t)$ implies the boundedness of $x_i(t)$. Taking into account the previous differential inequality and the equations for x_i in the system (5), we can get a number $T_1 = T_1(\varepsilon, P_0)$, $P_0 = (S_0, x_{10}, x_{20}, y_{10}, y_{20})$, such that

$$0 < S(t) < K + \varepsilon, 0 < x_i(t) < M_i + \frac{\varepsilon}{pD_i}, 0 < y_i(t) < \frac{K}{p} + 1 + \frac{\varepsilon}{p}, \tag{7}$$

where $M_i = \dfrac{K}{pD_i} + \dfrac{1}{D_i} + 1$, for all $t > T_1$.Thus we have the pointwise dissipativity for the systems (5).

The following corollary give us the existence of an global attractor.

Corollary 3. *The solution of the system 1 have a global attractor.*

Proof. It is an inmediate consequence of the theorem 3.4.6, page 39, in [4].

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