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Approximation of attractors for multivalued random dynamical systems

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Abstract. The concept of global attractor for stochastic partial differential inclusions has been recently introduced as a joint generalization of the theory of random attractors for random dynamical systems and global attractors for multivalued semiflows. We present a general result on the upper semicontinuity of attractors for multivalued random dynamical systems. In particular, our theory shows how the random attractor associated to a small random perturbation of a (deterministic) partial differential inclusion approximates the global attractor of the limiting problem. Some applications ilustrate the results.

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1 Introduction

When a phenomenon from Physics, Chemistry, Biology, Economics can be modelled by a system of differential equations where the existence of global solutions can be assured, one of the most interesting problems is to know what is the asymptotic behaviour of the system when time grows to infinite. In this context, the concept of *global attractor* has become a very useful tool to describe the long-time

This is an overview article.

behaviour of many important differential equations (Ladyzhenkaya [13], Babin and Vishik [4], Hale [12], Temam [16]). Most of this theory has been successfully and deeply developed for autonomous deterministic partial differential equations.

Some difficulties appear when we have to work without uniqueness of solutions in the system or when the model is better described by, for instance, a differential inclusion. In these cases, it has been shown that the theory of *multivalued flows* makes suitable the treatment of the asymptotic behaviour of these differential equations and inclusions, and the concept of global attractor can be generalized to this situation (Ball [5], Melnik and Valero [14]).

When a random term is added to the deterministic equation, the corresponding stochastic partial differential equation must be treated in a different way. The new and rapidly growing theory of *random dynamical systems* (Arnold [1]) has become the appropriate tool for the study of many important random and stochastic differential equations. In this framework, Crauel and Flandoli [11] (see also Schmalfuss [15]) introduced the concept of random attractor as a proper generalization (see Caraballo et al. [7] for a justification of this fact) of the corresponding (deterministic) global attractor.

Recently, Caraballo et al. [8,9] have generalized the theory of attractors to the case of stochastic partial differential inclusions. A global attractor in this case is a family of compact sets, invariant for the corresponding multivalued random dynamical system and attracting all bounded sets "from $-\infty$" at any fixed final time (see Section 2). The theory has been successfully applied to some stochastic differential inclusions with additive and multiplicative noise.

We study in this note the relationship between the random attractor for multivalued random dynamical systems and the global attractor for (deterministic) multivalued semiflows. Indeed, given a multivalued semiflow with an associated global attractor, we can add a small random perturbation so that the new stochastic multivalued dynamical system has a random attractor. Although the two attracting sets are too different (a compact set for the deterministic situation and an unbounded familiy of sets in the random case), we are able to prove a result on the upper semicontinuity of the attractors when we make the random perturbation tend to zero. In other words, when the perturbations are small enough, each compact set of the random attractor is within a small neighbourhood of the global attractor associated to the multivalued semiflow. This result, together with that in Caraballo et al. [7], in our opinion, comes to reinforce the concept of random attractor as a proper generalization of global attractors for deterministic differential equations and inclusions. We apply this result to a stochastic partial differential inclusion generated by additive white noise.

2 Multivalued dynamical systems (MDS)

In this section, we summarize the main concepts and results on dynamical systems related to problems containing multivalued functions and stochastic terms. These situations have led to different branches in the theory of dynamical systems and they have become the proper frameworks for the study of their qualitative properties.

2.1 Multivalued semiflows and global attractors

Let (X, d_X) be a complete and separable metric space with the Borel σ-algebra $\mathcal{B}(X)$, $\mathbb{R}_+ = [0, +\infty)$ and $P(X)$, $B(X)$, $C(X)$, $K(X)$ be the set of all nonempty, nonempty bounded, nonempty closed and nonempty compact subsets of X, respectively.

Definition 1. The map $G : \mathbb{R}_+ \times X \to P(X)$ is called a multivalued semiflow if it satisfies:

i) $G(0, \cdot) = I$ (the identity map in X).
ii) $G(t+s)x \subseteq G(t)G(s)x$, for all $t, s \in \mathbb{R}_+$, $x \in X$, where $G(t)D = \bigcup_{d \in D} G(t)d$, $D \subset X$.

Definition 2. Given a multivalued semiflow G and $A, B \subset X$, it is said that A attracts B with respect to G if

$$dist(G(t)B, A) \to 0 \text{ as } t \to +\infty,$$

where '$dist$' denotes the Hausdorff semidistance defined by

$$dist(C, D) = \sup_{c \in C} \inf_{d \in D} d(c, d).$$

The set A is said to be attracting for G if it attracts every $B \in B(X)$.

Definition 3. A set $\mathcal{A} \subset X$ is called a global attractor associated to G if it is attracting and negatively semi-invariant, that is, $\mathcal{A} \subset G(t)\mathcal{A}$, for all $t \in \mathbb{R}_+$.

Remark 4. In some applications, the global attractor will be a compact subset of the phase space X and invariant for the semiflow (i.e. $\mathcal{A} = G(t)\mathcal{A}$ for all $t \in \mathbb{R}_+$) (Melnik and Valero [14]). Note that this concept is similar to that of global attractor for single-valued semiflows (or dynamical systems) associated, for example, to partial differential equations (Hale [12], Temam [16]).

For the sake of completeness, we shall recall the definitions of upper and lower semicontinuity.

Definition 5. $G(t)$ is said to be *upper semicontinuous* if given $x \in X$ and a neighbourhood of $G(t)x$, $\mathcal{O}(G(t)x)$, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$ then

$$G(t)y \subset \mathcal{O}(G(t)x).$$

On the other hand, $G(t)$ is called *lower semicontinuous* if given $x_n \to x$ $(n \to +\infty)$ and $y \in G(t)x$, there exists $y_n \in G(t)x_n$ such that $y_n \to y$.

It is said to be *continuous* if it is upper and lower semicontinuous.

Remark 6. Note that these two definitions are not equivalent in general, as can be seen in easy examples (Aubin and Frankowska [3], Section 1.4).

Definition 7. $G(t)$ is said to be w-*upper semicontinuous* if for all $x_0 \in X$ we have that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$dist\,(G\,(t)\,y, G\,(t)\,x_0) \leq \varepsilon,$$

for any y satisfying $d_X(y, x_0) \leq \delta(\varepsilon)$.

Remark 8. Obviously, any upper semicontinuous map is w-upper semicontinuous. The converse is true when $G(t)$ has compact values (see Aubin & Cellina [2]).

The following is a general result for the existence of global attractors for multivalued semiflows (see Melnik and Valero [14] and also Ball [5] for a similar result):

Theorem 9. *Let $G(t) : X \to C(X)$ be an upper semicontinuous map, for all $t \in \mathbb{R}_+$. Suppose there exists a compact attracting set $K \subset X$. Then, there exists a global compact attractor $\mathcal{A}$. Moreover, it is the minimal closed attracting set.*

2.2 Multivalued random dynamical systems (MRDS) and random attractors

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$ a measure preserving group of transformations in Ω such that the map $(t, \omega) \mapsto \theta_t\omega$ is measurable and satisfying

$$\theta_{t+s} = \theta_t \circ \theta_s = \theta_s \circ \theta_t; \qquad \theta_0 = Id.$$

The parameter t takes now values in $\mathbb{R}$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Definition 10. A set valued map $G : \mathbb{R}_+ \times \Omega \times X \to C(X)$ is called a multivalued random dynamical system (MRDS) if it is measurable (that is, the inverse image of an open set is measurable; see Aubin and Frankowska [3, Definition 8.1.1]) and satisfies:

i) $G(0, \omega) = Id$ *on* X;
ii) $G(t+s, \omega)x = G(t, \theta_s\omega)G(s, \omega)x$, $\forall t, s \in \mathbb{R}^+, x \in X$ and $\omega \in \Omega$ *(perfect cocycle property)* .

Remark 11. We note that throughout this contribution all the results are obtained for ω in a θ-invariant subset of Ω of probability one, which does not depend on the time variable t. In order to avoid any confusion we shall write "for all $\omega \in \Omega$" instead of "for $\mathbb{P}$-a.a." when the time variable appears.

Definition 12. A *closed random set* D is a map $D : \Omega \to C(X)$, which is measurable. The measurability must be understood in the sense of Castaing and Valadier [10] for measurable multifunctions, that is, $\{D(\omega)\}_{\omega\in\Omega}$ is measurable if given $x \in X$ the map

$$\omega \in \Omega \mapsto dist(x, D(\omega))$$

is measurable, where $dist(x, D(\omega)) = \inf_{d\in D(\omega)} d_X(x,d)$.

A closed random set $D(\omega)$ is said to be *negatively (resp. strictly) invariant* for the MRDS if

$$D(\theta_t\omega) \subset G(t,\omega)D(\omega) \text{ (resp. } D(\theta_t\omega) = G(t,\omega)D(\omega)),\ \forall t \in \mathbb{R}^+, \omega \in \Omega.$$

Remark 13. This concept of measurability and the previous one are equivalent (see Aubin and Frankowska [3, Theorem 8.3.1]).

Suppose the following conditions for the MRDS G:

(G1) There exists an absorbing random compact set $B(\omega)$, that is, for every bounded set $D \subset X$, there exists $t_D(\omega)$ such that for all $t \geq t_D(\omega)$ one has

$$G(t, \theta_{-t}\omega)D \subset B(\omega),\ \mathbb{P}-a.s. \tag{1}$$

(G2) $G(t,\omega) : X \to C(X)$ is upper semicontinuous, for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Definition 14. The closed random set $\omega \mapsto \mathcal{A}(\omega)$ is said to be a global random attractor for the MRDS G if:

i) $G(t,\omega)\mathcal{A}(\omega) \supseteq \mathcal{A}(\theta_t\omega)$, for all $t \geq 0$, $\omega \in \Omega$ (that is, it is negatively invariant);

ii) For all bounded $D \subset X$,

$$\lim_{t\to+\infty} \operatorname{dist}(G(t, \theta_{-t}\omega)D, \mathcal{A}(\omega)) = 0, \mathbb{P}-a.s.;$$

iii) $\mathcal{A}(\omega)$ is compact $\mathbb{P}-a.s.$

We have the following theorem on the existence of random attractors for MRDS (Caraballo et. al. [8]):

Theorem 15. *Let (G1)–(G2) hold, the map $(t,\omega) \longmapsto \overline{G(t,\omega)D}$ be measurable, for all $D \in B(X)$, and the map $(t,\omega,x) \mapsto G(t,\omega)x$ have compact values. Then there exists the global random attractor $\mathcal{A}(\omega)$ for G, strictly invariant and measurable with respect to $\mathcal{F}$. It is unique and the minimal closed attracting set.*

3 Upper semicontinuity of attractors

Suppose that we have a deterministic multivalued dynamical system $G_0 : \mathbb{R}_+ \times X \to P(X)$ such that it satisfies the conditions of Theorem 9, so that there exists a global attractor $\mathcal{A}$ associated to it. Assume also that G_0 has compact values. Now we perturb the multivalued system by adding a random term depending on a small parameter $\sigma \in (0,1]$, so that we obtain a MRDS

$$G_\sigma : \mathbb{R}_+ \times \Omega \times X \to C(X).$$

We shall assume the following condition:

(H1) For all $\omega \in \Omega$, $t \in \mathbb{R}^+$ it holds

$$dist(G_\sigma(t,\omega)x, G_0(t)x\ ) \to 0, \text{ as } \sigma \searrow 0,$$

uniformly on compact sets of X.

We also assume that for all $\sigma \in (0,1]$ G_σ satisfies (G1)–(G2), so that there exists a random attractor $\mathcal{A}_\sigma(\omega) \subset K_\sigma(\omega)$, where $K_\sigma(\omega)$ is a compact absorbing set for G_σ, and that the following condition holds:

(H2) There exists a compact set $K \subset X$ such that

$$\lim_{\sigma \searrow 0} dist(K_\sigma(\omega), K) = 0, \text{ for all } \omega \in \Omega.$$

Then we can prove the following result:

Theorem 16. *Assume that* (G1)–(G2), (H1)–(H2) *hold and* G_0 *satisifes the conditions of Theorem* 9 *and has compact values. Then,*

$$\lim_{\sigma \searrow 0} dist(\mathcal{A}_\sigma(\omega), \mathcal{A}) = 0, \quad \textit{for all } \omega \in \Omega.$$

4 Applications: a partial differential inclusion perturbed by additive noise

Let X be a real separable Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Consider the following stochastic differential inclusion which can be regarded as a random perturbation of a deterministic differential inclusion with the small parameter $\sigma \geq 0$,

$$\begin{cases} \dfrac{du}{dt} \in Au\left(t\right) + F\left(u\left(t\right)\right) + \sigma \sum_{i=1}^{m} \phi_i \frac{dw_i(t)}{dt}, \ t \in \left(0, T\right), \\ u\left(0\right) = u_0, \end{cases} \tag{2}$$

where $A : D\left(A\right) \to X$ is a linear operator, $\phi_i \in D\left(A\right)$ and $w_i\left(t\right)$ are independent two-sided (i.e. $t \in \mathbb{R}$) real Wiener processes with $w_i\left(0\right) = 0$, $i = 1, ..., m$.

Let us introduce the following conditions:

(A) The operator A is m-dissipative, i.e.

$$\langle Ay, y \rangle \leq 0\,, \forall y \in D(A),$$

and $Im(A - \lambda I) = X$, $\forall \lambda > 0$.

(F1) $F : X \to C_v(X)$, where $C_v(X)$ is the set of all non-empty, bounded, closed, convex subsets of X.

(F2) The map F is Lipschitz on $\overline{D(A)}$, i.e. $\exists C \geq 0$ such that

$$dist_H(F(y_1), F(y_2)) \leq C\left\|y_1 - y_2\right\|, \forall y_1,\, y_2 \in \overline{D(A)},$$

where $dist_H(\cdot, \cdot)$ denotes the Hausdorff metric of bounded sets, i.e. it holds $dist_H(A, B) = \max\{dist(A, B), dist(B, A)\}$.

It is well known that for $\sigma = 0$, under some additional assumptions, differential inclusion (2) generates a multivalued semiflow G_0 which has the global compact attractor $\mathcal{A}$ (see Melnik & Valero [14]). Now we consider the random perturbation for $\sigma > 0$ and prove that we can construct a multivalued random dynamical system having a random attractor $\mathcal{A}_\sigma(\omega)$ which approximates the deterministic one in the upper semicontinuity sense.

Let us denote $\zeta(t) = \sum_{i=1}^m \phi_i w_i(t)$ and consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \{\omega = (w_1(\cdot), ..., w_m(\cdot)) \in C(\mathbb{R}, \mathbb{R}^m) \mid \omega(0) = 0\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$ and the Wiener measure $\mathbb{P}$. Each $\omega \in \Omega$ generates a map $\zeta(\cdot) = \sum_{i=1}^m \phi_i w_i(\cdot) \in C(\mathbb{R}, X)$ such that $\zeta(0) = 0$.

We make the change of variable $v(t) = u(t) - \sigma\zeta(t)$. Inclusion (2) turns into

$$\begin{cases} \dfrac{dv}{dt} \in Av(t) + F(v(t) + \sigma\zeta(t)) + \sigma \sum_{i=1}^m A\phi_i w_i(t), \\ v(0) = v_0 = u_0. \end{cases} \tag{3}$$

We shall define the multivalued map $\widetilde{F}_\sigma : [0, T] \times \Omega \times X \to C_v(X)$,

$$\widetilde{F_\sigma}(t, \omega, x) = F(x + \sigma\zeta(t)) + \sigma A\zeta(t).$$

It is easy to obtain from (F2) the existence of constants D_1, D_2 such that

$$\|F(x)\|^+ \le D_1 + D_2 \|x\|, \tag{4}$$

where $\|F(x)\|^+ = \sup\limits_{y \in F(x)} \|y\|$. Hence,

$$\left\|\widetilde{F_\sigma}(t, \omega, x)\right\|^+ \le D_1 + D_2\|x\| + D_2\sigma\|\zeta(t)\| + \sigma\|A\zeta(t)\| = n_\sigma(t, \omega, x). \tag{5}$$

It follows that $\widetilde{F_\sigma}$ satisfies the next property:

(F3) For any $x \in X$ there exists $n(\cdot) \in L_1(0, T)$ depending on x, ω and σ such that

$$\left\|\widetilde{F}_\sigma(t, \omega, x)\right\|^+ \le n_\sigma(t), \text{ a.e. in } (0, T).$$

On the other hand, it is clear that $\widetilde{F_\sigma}$ satisfies conditions (F1)–(F2) for any fixed $t \in [0, T]$ and $\omega \in \Omega$, where the constant C does not depend on t or ω or σ.

Consider also the equation

$$\begin{cases} \dfrac{dv(t)}{dt} = Av(t) + f(t), \\ v(0) = v_0, \end{cases} \tag{6}$$

where $f(\cdot) \in L_1([0, T], X)$.

Definition 17. The continuous function $v : [0,T] \to X$ is called an integral solution of problem (6) if:

1. $v(0) = v_0$;
2. $\forall \xi \in D(A)$, one has

$$\|v(t) - \xi\|^2 \le \|v(s) - \xi\|^2 + 2\int_s^t \langle f(\tau) + A\xi, v(\tau) - \xi\rangle \, d\tau, \ t \ge s. \quad (7)$$

Definition 18. The function $u : [0,T] \to X$ is called a strong solution of problem (6) if:

1. $u(\cdot)$ is continuous on $[0,T]$ and $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on any compact subset of $(0,T)$ and almost everywhere (a.e.) differentiable on $(0,T)$;
3. $u(\cdot)$ satisfies (6) a.e. on $(0,T)$.

It is well known (see Barbu [6, p.124]) that any strong solution of problem (6) is an integral solution.

Definition 19. The function $v_\sigma : [0,T] \times \Omega \to X$ is said to be an integral solution of problem (3) if for any $\omega \in \Omega$ one has:

1. $v_\sigma(\cdot) = v_\sigma(\cdot,\omega) : [0,T] \to X$ *is continuous.*
2. $v_\sigma(0) = v_0$;
3. *For some selection* $f \in L_1([0,T],X)$, $f(t) \in \widetilde{F_\sigma}(t,\omega,v_\sigma(t))$ *a.e. on* $(0,T)$, *inequality* (7) *holds.*

If condition (A) holds and $f \in L_1([0,T],X)$, then for all $v_0 \in \overline{D(A)}$ there exists a unique integral solution $v(\cdot)$ of (6) for each $T > 0$ (see Barbu [6, p.124]). We shall denote this solution by $v(\cdot) = I(v_0)f(\cdot)$. Since $T > 0$ is arbitrary, each solution can be extended on $[0,\infty)$. Let us denote by $\mathcal{D}_\sigma(v_0,\omega)$ the set of all integral solutions of (3) such that $v(0) = v_0$ ($\mathcal{D}_0(v_0)$ for $\sigma = 0$). We define the maps $G_\sigma : \mathbb{R}_+ \times \Omega \times \overline{D(A)} \to P(\overline{D(A)})$, $\vartheta_s : \Omega \to \Omega$ as follows

$$G_\sigma(t,\omega)v_0 = \{v_\sigma(t) + \sigma\zeta\,(t) \mid v_\sigma(\cdot) \in \mathcal{D}_\sigma(v_0,\omega)\},$$

$$\theta_s\omega = (w_1\,(s+\cdot) - w_1\,(s)\,, ..., w_m\,(s+\cdot) - w_m\,(s)) \in \Omega.$$

Then the function $\widetilde{w}$ corresponding to $\theta_s\omega$ is defined by $\widetilde{\zeta}\,(\tau) = \zeta\,(s+\tau) - \zeta\,(s) = \sum_{i=1}^m \phi_i\,(w_i\,(s+\tau) - w_i\,(s))\,.$

By similar computations to the ones in Caraballo et al. [8] but taking care of the new parameter σ, we have the following results:

Theorem 20. *Let* (A), (F1), (F2) *hold and the semigroup* $S\,(t,\cdot)$ *generated by the operator* A *be compact. Then,* G_σ *generates a MRDS.*

Proposition 21. *Let* (A), (F1), (F2) *hold. Suppose that each integral solution of* (3), $v_\sigma(\cdot) = I(u_0) f(\cdot)$ *is a strong solution of* (6). *Let there exist constants* $\delta > 0$, $M \geq 0$ *such that*

$$\langle y, u \rangle \leq (-\delta + \alpha) \|u\|^2 + M, \forall u \in D(A), y \in F(u), \tag{8}$$

where $\alpha \geq 0$ *is the biggest constant such that*

$$\langle Au, u \rangle \leq -\alpha \|u\|^2. \tag{9}$$

Then there exists a random radius $r_\sigma(\omega) > 0$ *such that for any bounded set* $B \subset \overline{D(A)}$ *we can find* $T(B) \leq -1$ *for which*

$$\|G_\sigma(-1 - t_0, \theta_{t_0}\omega) u_0\|^+ \leq r_\sigma(\theta_{-1}\omega), \mathbb{P} - a.s., \forall t_0 \leq T(B), \forall u_0 \in B.$$

Theorem 22. *Let the conditions of Proposition* 21 *hold, the semigroup* $S(t, \cdot)$ *generated by the operator* A *be compact and the multivalued map* $G_\sigma(1, \omega)$ *be* $\mathbb{P}$*-a.s. compact (that is, it maps bounded sets into precompact ones). Then* G_σ *satisfies* (G1)–(G2) *and has the minimal global random attractor* $\mathcal{A}_\sigma(\omega)$. *Moreover, it is invariant and measurable with respect to* $\mathcal{F}$.

Remark 23. Note that in the case where $\sigma = 0$ the semiflow G_0 satisfies the conditions of Theorem 9, the global attractor obtained in Theorem 22 is deterministic and coincides with the global attractor $\mathcal{A}$ of Melnik & Valero [14].

Proposition 24. *In the conditions of Theorem* 22 *we have that*

$$\lim_{\sigma \searrow 0} dist(G_\sigma(t, \omega)x, G_0(t)x) = 0, \text{ for all } \omega \in \Omega, t \in \mathbb{R}_+,$$

uniformly on compact subsets of $\overline{D(A)}$.

For (H2) note that a compact absorbing set for G_σ is given by

$$K_\sigma(\omega) = \overline{G_\sigma(1, \theta_{-1}\omega) B(r_\sigma(\theta_{-1}\omega))},$$

where $B(r_\sigma(\theta_{-1}\omega))$ is a ball in H with radius $r_\sigma(\theta_{-1}\omega)$ (see Caraballo et. al. [8]).

Lemma 25. *In the conditions of Theorem* 22 *we have for the compact set* $K = \overline{G_0(1, B(R))}$

$$dist(K_\sigma(\omega), K) \to 0, \text{ as } \sigma \searrow 0, \ \mathbb{P}\text{-}a.s.$$

We have proved that (H2) holds.

As a consequence of Theorem 22, Proposition 24, Lemma 25 and Theorem 16 we obtain:

Theorem 26. *In the conditions of Theorem* 22 *we have*

$$dist(\mathcal{A}_\sigma(\omega), \mathcal{A}) \to 0, \text{ as } \sigma \searrow 0, \mathbb{P}\text{-}a.s.$$

Remark 27. The theory can also be applied to differential inclusions perturbed by multiplicative noise (Caraballo et al. [9]).

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