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Colouring polytopic partitions in $\mathbb{R}^d$

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Abstract. We consider face-to-face partitions of bounded polytopes into convex polytopes in $\mathbb{R}^d$ for arbitrary $d \geq 1$ and examine their colourability. In particular, we prove that the chromatic number of any simplicial partition does not exceed $d+1$. Partitions of polyhedra in $\mathbb{R}^3$ into pentahedra and hexahedra are 5- and 6-colourable, respectively. We show that the above numbers are attainable, i.e., in general, they cannot be reduced.

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1 Introduction

In 1890, P. J. Heawood formulated his famous map-colouring theorem (see [10]), which determines an attainable upper bound of the chromatic number of maps on two-dimensional compact orientable surfaces whose genus is positive. The case of genus 0 (known as the four colour conjecture for planar maps) was for a long time an open problem and served as a catalyst for graph theory. In 1930, Kasimir Kuratowski introduced his well-known necessary and sufficient condition for testing the planarity of a graph, see [14]. (An algorithm for testing the planarity can be found, e.g., in [17].)

It was not until in 1976 that K. Appel and W. Haken proved with the help of computers that every planar map is 4-colourable (see [1], [2], [3]). A simpler

This is author's version of the invited lecture.

proof, which is also based on the use of computers, is given in [16]. Recall that the colouring of maps and graphs has a lot of practical applications (storing chemical compounds, designing optimal time-tables, allocating frequencies for mobile phones, etc.).

Let us consider now a three-dimensional "map", i.e., a partition of a three-dimensional bounded region into a finite number of subregions.

Does there exist an analogue of the four colour theorem in $\mathbb{R}^3$*?*

In Figure 1 we see a simple example showing nonconvex three-dimensional subregions each of which touches all the others. Such regions can be modelled by L-shaped flexible pieces of paper with positive thickness (this can obviously also be done by polyhedra). The configuration of Figure 1 can be associated with a graph whose vertices correspond to regions and such that two vertices are joined by an edge whenever the corresponding regions are adjacent. For n such subregions we obtain the complete graph K_n, which requires n different colours. It is obvious that the number of such subregions, and therefore also the number of colours, can be arbitrarily large (see the last column of Table 1 in Section 4).

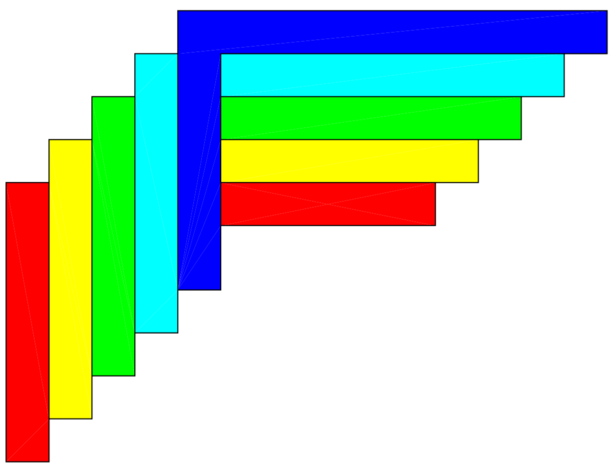

Figure 1

In this paper we show that, if we allow only maps with convex subregions, we might expect that there exists for each $d \in \{1, 2, 3, \dots\}$ a fixed finite upper bound for the "chromatic number" for arbitrarily many d-dimensional subregions. Figure 1 thus illustrates that the assumption of convexity is essential for $d \geq 3$. Since according to [18, p. 902], the only convex compact sets that tile the space $\mathbb{R}^d$ are convex polytopes, we shall from now on consider only subregions that are compact convex polytopes.

With the terminology of the finite element method in mind, we will call any compact convex polytope in $\mathbb{R}^d$, $d = 1, 2, 3, \dots$, whose interior is nonempty, an *element*. Its $(d-1)$-dimensional faces will for simplicity be called *faces*.

Let $\Omega \subset \mathbb{R}^d$ be a bounded polytopic domain and denote its boundary by $\partial\Omega$. We shall only consider face-to-face partitions of $\overline{\Omega}$ into convex d-dimensional polytopes (the main reason for this assumption is given in Remark 4.1).

A finite set $\mathcal{T}$ of elements is said to be a *partition* of $\overline{\Omega}$ into elements if

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}} T, \tag{1.1}$$

if the interiors of any two elements from $\mathcal{T}$ are disjoint, and if any face of any element $T \in \mathcal{T}$ is either a subset of the boundary $\partial\Omega$, or a face of another element in the partition. Two elements are called *adjacent* if they have a common face.

One of the most important features of the finite element method for solving three-dimensional boundary value problems on a bounded polyhedral domain Ω is the generation of a partition of $\overline{\Omega}$ (see [12]) into elements. The existence of such a partition into tetrahedra for an arbitrary bounded polyhedral domain is given in [11]. The visualization of such a three-dimensional partition into tetrahedra, pentahedra (pyramids, triangular prisms), hexahedra, etc., is an important and difficult problem. One way is to paint adjacent elements with different colours. Also in two-dimensional space, elements are often coloured to emphasize their positions in the triangulation considered (see, e.g., [4]). We meet a similar problem in domain decomposition methods, where adjacent subdomains are painted with different colours to emphasize their positions. Moreover, there are fast iteration methods that perform calculations on subdomains with the same colour in parallel processors (see, e.g., [19]).

We now highlight several standard definitions from graph theory. A *colouring* of a partition $\mathcal{T}$ is an assignment of colours to its elements such that no two adjacent elements have the same colour. An *n-colouring* of a partition $\mathcal{T}$ uses n colours. A partition is said to be *n-colourable* if there exists a colouring of $\mathcal{T}$ that uses n colours or fewer. The *chromatic number* $\chi(\mathcal{T})$ is defined as the minimum n for which $\mathcal{T}$ has an n-colouring.

So, we stress that a partition $\mathcal{T}$ is n-chromatic if $\chi(\mathcal{T}) = n$, and n-colourable if $\chi(\mathcal{T}) \le n$. Throughout the paper, colours will for convenience be denoted by the numbers $1, 2, \ldots, n$.

One of the aims of this paper is to prove that for any simplicial partition in $\mathbb{R}^d$ there exists a $(d+1)$-colouring. We start with two-dimensional partitions into triangles just to introduce the main idea of the proof of the general result, Theorem 3.3.

2 Colouring triangulations

By a *triangulation* we mean a (face-to-face) partition of a bounded polygon $\overline{\Omega} \subset \mathbb{R}^2$ into (closed) triangles.

The famous PLTMG program (see [4]) for solving partial differential equations generates triangulations of $\overline{\Omega}$, which are coloured with 5 different colours such that any two adjacent triangles have different colours. According to the four colour theorem, this number could clearly have been reduced to 4.

Remark 2.1. In contrast to the colouring of a general map, it is very easy to find an algorithm for a 4-colouring of any triangulation. We can proceed, for instance, by induction. Assume we have a map (triangulation) with k triangles. Remove an arbitrary triangle and assign a colouring to the remaining map of $k-1$ triangles. Then add the kth triangle again and colour it differently than its (max. 3) neighbours.

By Brooks' theorem (see, e.g., [15]), if G is a graph with maximum degree $n \geq 3$ and if G does not contain the complete graph K_{n+1}, then G is n-colourable. Proposition 2.2 below is a special case of Brooks' theorem with $n = 3$. However, its proof differs from the one presented in [15] and is constructive, i.e., it can be used as a colouring algorithm. We show that the number of colours can be reduced to 3 for any triangulation (cf. Figure 2). The key point is the avoidance of colourings containing a triangle surrounded by three triangles already coloured with three different colours.

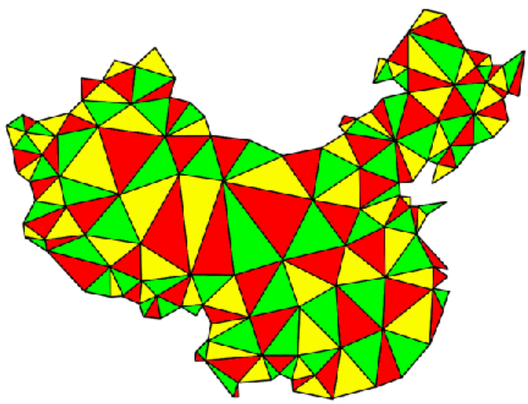

Figure 2

Proposition 2.2. *Any triangulation is* 3*-colourable.*

Proof. Let $\mathcal{T}$ be a triangulation of a bounded polygon $\overline{\Omega}$ consisting of k triangles (cf. (1.1)). First, number the triangles inductively as follows. Let $M_1 = \Omega$ and let i successively increase from 1 to k. Choose an arbitrary $T_i \in \mathcal{T}$ which has at least one side on the boundary ∂M_i and then set

$$M_{i+1} = M_i \setminus T_i.$$

We observe that $\overline{M_k} = T_k$.

Second, let i successively decrease from k to 1. Since each T_i has at most two neighbours with higher indices, we may assign to T_i any colour different from its at most two neighbours.

In detail, we define the colour $c(T_i)$ of the ith triangle T_i, for instance, by

$$c(T_i) = \min(B_i), \tag{2.1}$$

where

$$B_i = \{1, 2, 3\} \setminus A_i,$$

and $A_i \subset \{1, 2, 3\}$ is the set of colours of those adjacent triangles of $T_i \subset \overline{M}_i$ that were already coloured.

Remark 2.3. The function min in (2.1) can be obviously replaced by max, or in applications by rnd which (pseudo)randomly chooses an element from the set B_i.

Remark 2.4. To any given triangulation we may associate, in a standard way, a graph whose nodes correspond to triangles and whose edges indicate that two triangles are adjacent. Since every triangle in the triangulation has at most three adjacent triangles, the degree of each node is at most 3. In Figure 3 we see a 4-colourable graph K_4 whose nodes all have degree 3. By the contrapositive of Proposition 2.2, this graph cannot correspond to any planar triangulation. Note that the surface of a ball can be decomposed into four "curved triangles", by projecting the regular tetrahedron from its centre of gravity into a circumscribed ball. The corresponding graph is, indeed, exactly the one given in Figure 3.

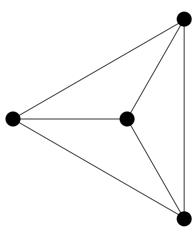

Figure 3

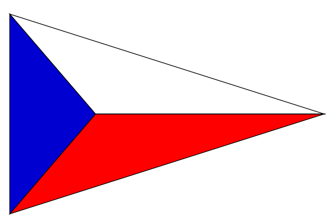

Figure 4

Remark 2.5. If some vertex in a triangulation is surrounded by an odd number of triangles, then the number of colours cannot be 2 (see Figure 4). On the other hand, standard periodic triangulations (uniform, chevron, criss-cross, union-jack) in finite-element theory, which yield various superconvergence phenomena [13], are 2 colourable – see Figure 5. This follows from the classical theorem which states that *a graph is 2-colourable if and only if it has no odd-length cycles* (see, e.g., [7, p. 37], [9, p. 127], [20, p. 235]).

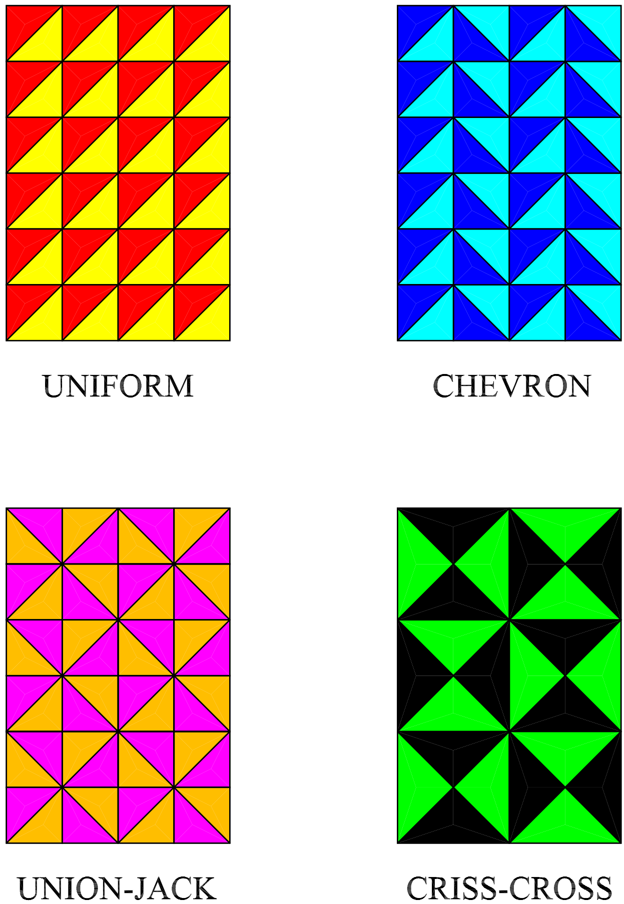

Figure 5

Remark 2.6. There are several theorems on 3-colouring. For instance, according to the well-known Grötzsch's theorem (see [8], [9, p. 131]), *every planar graph with fewer than* 4 *"triangles" is* 3*-colourable.* (Here the word "triangle" has to be understood in the context of graph theory.) Note that the graph in Figure 3 has 4 "triangles". It is obvious that Proposition 2.2 does not follow from Grötzsch's theorem, since there exist triangulations in which 4 different vertices are surrounded by three triangles (see Figure 2, for example).

3 Colouring polytopic partitions in $\mathbb{R}^d$

In this section we generalize Proposition 2.2 to $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$, and to arbitrary elements (i.e., to compact convex polytopes whose interior is nonempty in $\mathbb{R}^d$).

Remark 3.1. The chromatic number of a planar partition into convex polygons is, in general, larger than the chromatic number of a triangulation. For instance, in Figure 6 we see a planar partition whose elements are not all triangles and whose chromatic number is 4 (the associated graph is in Figure 3). Similarly, for partitions in $\mathbb{R}^d$ we need, in general, more colours if the number of faces of each element is greater than the number of faces of a d-simplex.

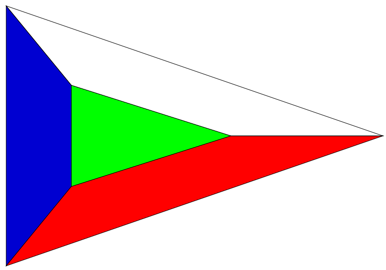

Figure 6

Remark 3.2. Assume that the number of faces of each element of a partition $\mathcal{T}$ in $\mathbb{R}^d$ does not exceed a given number f. Clearly,

$$f > d,$$

since any d-simplex has $d + 1$ faces. A simple algorithm for a $(f + 1)$-colouring of any such partition is as follows: We assign one of the $f + 1$ colours to each element in turn, giving each element a colour not already assigned to any adjacent element.

The next theorem shows that the number of colours can be reduced to f.

Theorem 3.3. *Let the number of faces of each polytope of a partition $\mathcal{T}$ in $\mathbb{R}^d$ not exceed a given number f. Then $\mathcal{T}$ is f-colourable.*

Proof. Let $\mathcal{T}$ be a partition with k elements. First, let i successively increase from 1 to k. We denote by $T_1 \in \mathcal{T}$ any element whose face lies on the boundary of Ω, by T_2 any element whose face lies on the boundary of $\Omega \setminus T_1$, by T_3 any element whose face lies on the boundary of $\Omega \setminus (T_1 \cup T_2)$, etc. In other words, $T_i \in \mathcal{T}$ is any element whose face lies on the boundary ∂M_i of the open set

$$M_i = \Omega \setminus \bigcup_{j=1}^{i-1} T_j \quad \text{for } i = 1, \ldots, k.$$

In particular $M_1 = \Omega$ and $\overline{M_k} = T_k$. We see that the boundary ∂M_i is nonempty for $i = 1, \ldots, k$.

Second, we shall colour elements contained in the set $\overline{M}_i$, where i successively decreases from k to 1. We set

$$c(T_i) = \min(B_i), \tag{3.1}$$

where

$$B_i = \{1, 2, \dots, f\} \setminus A_i$$

and $A_i \subset \{1, 2, \dots, f\}$ is the set of colours of those adjacent elements of $T_i \subset \overline{M}_i$ that were already coloured.

Further, we have to show that B_i is nonempty to guarantee that the colour $c(T_i)$ in (3.1) is well defined. Since T_i has at least one face in $\partial M_i (\neq \emptyset)$, the element T_i has at most $f - 1$ adjacent elements in the set M_i, and thus the cardinality of the set A_i is at most $f - 1$. Consequently, B_i is nonempty and $c(T_i)$ is correctly defined.

For a better visualization, elements in three-dimensional partitions will be usually illustrated in "exploded configurations" in which they do not touch its neighbouring elements.

Theorem 3.4. *Any simplicial partition in* $\mathbb{R}^d$ *is* $(d+1)$*-colourable and this number cannot, in general, be reduced.*

Proof. Any d-simplex has $d + 1$ faces $F_1, F_2, \dots, F_d, F_{d+1}$. Thus the first part of the theorem follows immediately from Theorem 3.3.

Now we show that there exists a simplicial partition $\mathcal{T}$ whose chromatic number is exactly $d+1$. Let T be an arbitrary d-simplex in $\mathbb{R}^d$ and let $P \in T$ be an arbitrary interior point (e.g., the center of gravity). Set $\mathcal{T} = \{T_i\}_{i=1}^{d+1}$, where

$$T_i = \operatorname{conv}(P, F_i) \quad \text{for } i = 1, 2, \dots, d + 1,$$

and where conv denotes the convex hull (see Figure 4 for $d = 2$ and Figure 7 for $d = 3$). Then each T_i is also a d-simplex in $\mathbb{R}^d$ and the chromatic number of $\mathcal{T}$ is exactly $d + 1$. This is because each T_i has d common faces with all remaining d-simplices T_j, $j \neq i$, whose number is d.

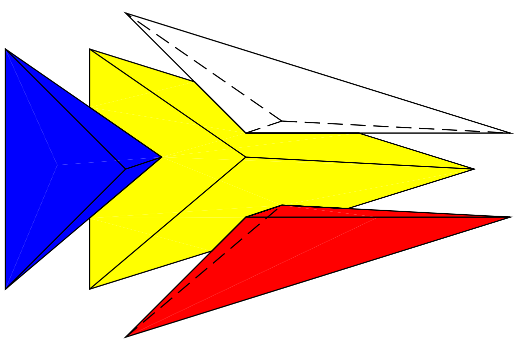

Figure 7

A partition in $\mathbb{R}^3$ consisting only of tetrahedra is called a *tetrahedralization*. A special case of Theorem 3.4 for $d = 3$ can be stated as follows:

Corollary 3.5 (The four colour theorem for tetrahedra in $\mathbb{R}^3$). *Any tetrahedralization is* 4*-colourable.*

Remark 3.6. Although any tetrahedralization is 4-colourable, the associated graph is not planar, in general. Thus Corollary 3.5 is not a consequence of the classical four colour theorem.

Remark 3.7. In Figure 8 we see an example of a uniform tetrahedralization which is only 2-colourable (cf. Remark 2.5) and whose associated graph is not planar.

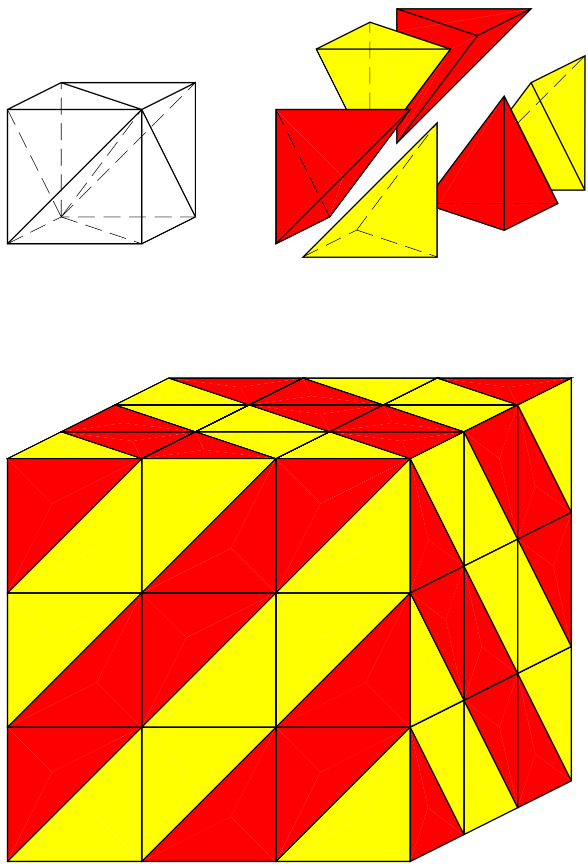

Figure 8

A partition in $\mathbb{R}^3$ into tetrahedra and pentahedra (pyramids, triangular prisms) is called a *pentahedralization*.

Theorem 3.8. *Any pentahedralization is* 5*-colourable and this number cannot be reduced, in general.*

Proof. By Theorem 3.3, the chromatic number of any pentahedralization is at most 5.

The construction of a pentahedralization $\mathcal{T}$ whose chromatic number is exactly 5 is sketched in Figure 9 (which represents a three-dimensional analogue of Figure 6). The pentahedralization $\mathcal{T}$ consists of a tetrahedron which is surrounded by 4 pentahedra such that each element touches all others. Therefore, the associated graph is K_5 and the chromatic number of $\mathcal{T}$ is exactly 5.

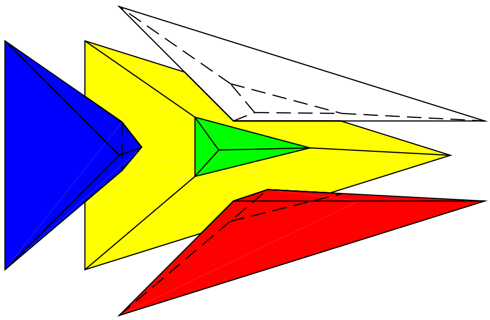

Figure 9

A partition in $\mathbb{R}^3$ all of whose elements (convex polyhedra) have at most 6 faces is called a *hexahedralization*.

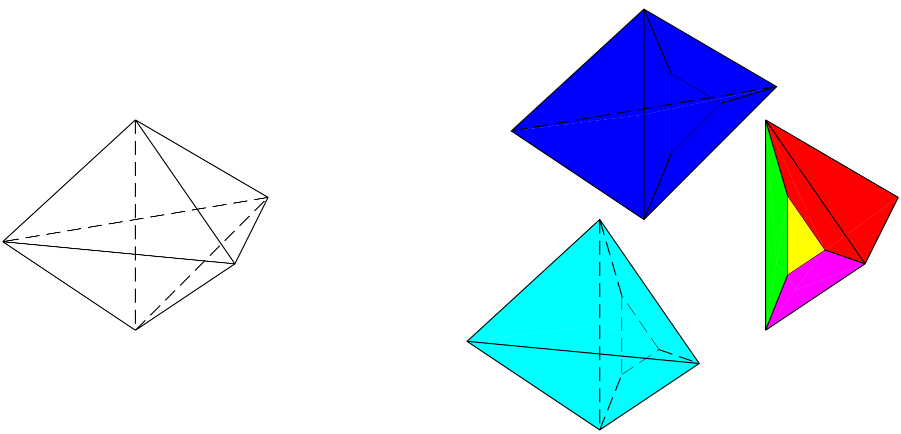

Figure 10

Theorem 3.9. *The chromatic number of any hexahedralization is at most* 6 *and there exists a hexahedralization whose chromatic number is exactly* 6*.*

Proof. The upper bound 6 is again given by Theorem 3.3. The lower bound 6 comes from the hexahedralization $\mathcal{T}$ marked in Figure 10. It shows a hexahedron (on the left), which is decomposed into 6 convex polyhedra (2 pentahedra and 4 hexahedra on the right) such that each one touches all the other polyhedra. Therefore, the associated graph is K_6 and the chromatic number of $\mathcal{T}$ is 6.

Remark 3.10. A partition similar to Figures 6 and 10 in $\mathbb{R}^d$, $d > 3$, can be constructed by induction. In this way, we obtain altogether $2d$ polytopes such that each one touches all others.

4 Endnotes and open problems

Remark 4.1. Figure 11 illustrates a decomposition of a triangular domain into 4 triangles, which is not face-to-face and whose associated graph is K_4. This example shows why we considered only face-to-face partitions. Note that finite element grids with the so-called hanging nodes require, in general, more colours than conforming grids (i.e., face-to-face partitions).

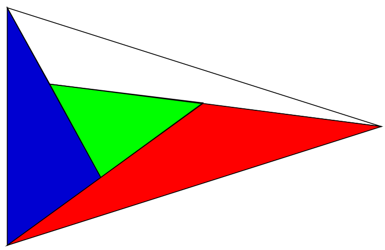

Figure 11

Colouring of subdomains can be applied in domain decomposition methods. When we use nonconforming mixed elements (see, e.g., [6]), which have no degrees of freedom at vertices (and edges for $d = 3$), then subdomains which have no common face, have no common degree of freedom. This enables us to compute the finite element solution on subdomains of the same colour simultaneously on parallel processors (cf. [19]).

Remark 4.2. We can prove, in the same way as Proposition 2.2, that any "triangulation" of the Möbius strip is 3-colourable.

Remark 4.3. Analogously to Remark 2.1, we can prove that the chromatic number of any "triangulation" of a torus (or a two-dimensional surface with a positive genus) is at most 4. The next example illustrates that this number cannot be reduced, in general. Consider a triangulation of a flexible piece of paper $ABCD$ as marked in Figure 12. We first glue up the segment AB with DC, and then AD with BC to obtain a triangulation of a torus whose associated graph is K_4. Let us still note that the surface of every toroidal polyhedron consisting of convex polygons is 6-colourable (see [5]).

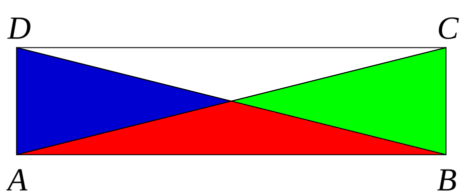

Figure 12

Remark 4.4. Standard finite elements used for solving three-dimensional problems have at most 6 faces (cf. Theorem 3.9). Consider now partitions in $\mathbb{R}^d$, where each element can have an arbitrary number of faces. In Table 1 we see the maximum chromatic numbers for any d. The numbers in the second column follow from Theorem 3.4. The symbol ? in the third column indicates that we know only a lower bound for the maximum chromatic number (see Theorem 3.9 and Remark 3.10). The upper bound of the maximum chromatic number is known only for $d \leq 2$ (cf. Figure 6). Finally, the last column corresponds to arbitrary regions, i.e., to connected domains that are nonconvex, in general (cf. Figure 1).

dimension	simplices	convex polytopes	arbitrary regions
1	2	2	2
2	3	4	4
3	4	6 ?	∞
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d	$d+1$	$2d$?	∞

Table 1. Maximum chromatic numbers for arbitrary partitions $\mathbb{R}^d$. The symbol ∞ means that the chromatic number can be arbitrarily large.

Remark 4.5. The numbers in the above table hold for infinite partitions of unbounded domains as well.

Conjecture 4.6. *Any partition of a polyhedron in $\mathbb{R}^3$ is 6-colourable.*

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