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Elliptic Equations with Decreasing Nonlinearity II: Radial Solutions for Singular Equations

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Abstract. By means of the super-sub-solutions method from [3], the existence of decreasing solutions of some singular elliptic equations will be established.

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1 Introduction

Let $f \in C^1([0,\infty);\mathbb{R}_+)$ with $f(r) > 0\ \forall r \geq 0$ and $f(r) \simeq r^{-\theta}$ at ∞ for some $\theta > 0$. For some $a > 1$ and $p \in (1,2]$, assume that

f) $\exists b \in (0, a+1-p]$; for $w(t) := (1+t)^{-b/(p-1)}$, some $\gamma > 0$ and

$$\psi(r) := f(r)w(r)^{-\gamma}, \qquad \int_0^\infty s^{b+p-1}\psi(s)ds < \infty.$$

In this note, we investigate the existence of positive and decreasing solutions $u \in C^2 := C^2([0,\infty))$ of

$$\left.\begin{aligned} Qu \equiv &(r^a|u'|^{p-2}u')' + r^a F_q^\nu(r,u)_+ = 0, \quad u'(0) = 0,\\ &\text{where}\quad \mathrm{q} > 0, \quad \mathrm{F}_\mathrm{q}^\nu(\mathrm{r},\mathrm{u}) := \mathrm{f}(\mathrm{r})\mathrm{u}^{-\gamma} - \nu \mathrm{u}^\mathrm{q}, \quad \nu \geq 0,\\ &\text{or}\quad \mathrm{F}_\mathrm{q}^\nu(\mathrm{r},\mathrm{u}) := \nu \mathrm{f}(\mathrm{r})\mathrm{u}^{-\gamma} + \mathrm{u}^\mathrm{q}, \quad \nu > 0. \end{aligned}\right\} \tag{Q}$$

For $a = n-1$, $n \in \mathbb{N}$, such u is a radial solution in $\mathbb{R}^n$ of the p-Laplacian equations $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + F_q^\nu(|x|,u)_+ = 0$.
For a positive and decreasing function ϕ, define

$$\Phi(r) = T\phi(r) := \phi(0) - \int_0^r dt\left\{\int_0^t (s/t)^a F_q^\nu(s,\phi)_+ ds\right\}^{1/(p-1)}.$$

This is the final form of the paper.

Given such a function ϕ, the following result from [3] will be used:

$$\text{assume that} \qquad \int_0^\infty (1+s^{p-1}) F_q^\nu(s,\phi)_+ ds < \infty; \tag{ϕ}$$

if $\forall r \geq 0 \quad Q\phi \geq 0 \quad (\leq 0$ *respectively) and* $F_q^\nu(r,.)$ *is positive and decreasing in* $[\Phi(r), \phi(r)]$ *(*$[\phi(r), \Phi(r)]$ *respect.), then* (Q) *has a decreasing solution* $u \in C^2([0,\infty))$ *such that* $\Phi \leq u \leq \phi \quad (\phi \leq u \leq \Phi$ *respect.) in* $[0,\infty)$*.*
The main results are the following:

Theorem 1 (Uniqueness). *Assume that* $\forall r \geq 0 \;\; t \mapsto F_q^\nu(r,t)_+$ *is decreasing in* $t > 0$*. Then*

a) $\forall b \geq 0$*, if it exists the decreasing solution* $u_b \in C^1$ *of* (Q) *such that* $\lim_\infty u_b = b$ *is unique;*
b) $\forall R > 0$*, if it exists the decreasing solution* $u \in C^1([0,R))$ *of* (Q) *such that* $u(R) = 0$ *is unique.*

Theorem 2 (Existence). *Suppose that for some* $\gamma > 0$ *and* $b \in (0, a+1-p]$

$$\int_0^\infty s^{b+p-1} f(s)(1+s)^{b\gamma/(p-1)} < \infty. \tag{1}$$

1) *Then, the equation*

$$(r^a |u'|^{p-2} u')' + r^a f(r) u(r)^{-\gamma} = 0 \tag{2}$$

has a unique positive and decreasing solution $u \in C^2 := C^2([0,\infty))$ *such that*

$$u \leq C\, r^{-b/(p-1)} \quad (u \simeq r^{-b/(p-1)} \text{ if } b = a+1-p) \text{ at } \infty;$$

2) *if also* $q > \max\{p(p-1)/b, -\gamma + \theta(p-1)/b\}$*,*
   i) *there is* $\nu_0 > 0$ *depending only on* f *such that for* $\nu \in (0, \nu_0]$

$$(r^a |v'|^{p-2} v')' + r^a \{f(r) v(r)^{-\gamma} - \nu v(r)^q\}_+ = 0 \tag{3}$$

   has a unique decreasing and positive solution $v \in C^2$*; if in addition* $q > (p-1)(b+p)/b$*, then* $v(r) \leq C\, r^{-b/(p-1)}$ *at* ∞*;*
   ii) *there is* $\nu_1 > 0$ *depending only on* f *such that* $\forall \nu > \nu_1$

$$(r^a |U'|^{p-2} U')' + r^a \{\nu f(r) U^{-\gamma} + U^q\} = 0 \tag{4}$$

   has a positive and decreasing solution U *such that* $U \leq C\, r^{-b/(p-1)}$ *at* ∞*.*

2 Preliminaries

Definitions and notations:
$\mu := 1/(p-1)$; $\quad m := \mu b$, $\quad b \in (0, a+1-p]$; $w(r) := (1+r)^{-m}$; $\int v(s) := \int v(s) ds$; $\psi(r) := f(r) w(r)^{-\gamma}$; $t_* := \max\{1, t\}$ and $D_a^p u := (r^a |u'|^{p-2} u')'$.

2.1 Properties of some integrals

Define for $t \geq 0$

$$J(t) := \int_t^\infty \left(\int_0^r \left(\frac{s}{r}\right)^a \psi(s) \right)^\mu . \tag{5}$$

We normalized f so that

$$\Psi_1 := \int_0^1 \left(\int_0^r \psi \right)^\mu + \frac{1}{m} \left(\int_0^\infty s^{b+p-1} \psi \right)^\mu \leq 1. \tag{6}$$

Lemma 3. *If*

$$\int_0^\infty s^{b+p-1}\psi(s) < \infty \qquad \text{or} \qquad 0 < \gamma < (p-1)\frac{(\theta - b - p)}{b}, \tag{7}$$

where $b \in (0, a+1-p]$, *then* $\forall t \geq 0$

$$\frac{(p-1)}{a+1-p} \left(\int_0^1 s^a \psi \right)^\mu \leq J(t) \leq \Psi_1 \, t_*^{-m}; \tag{8}$$

$$b = a+1-p \implies mJ(t) \geq t^{-m} \left\{ \int_0^1 s^a \psi(s) ds \right\}^\mu \quad \forall t > 1; \tag{9}$$

$$|J(t)'| \leq \left\{ \left(\int_0^1 \psi \right)^\mu + \left(\int_0^\infty s^{b+p-1} \psi \right)^\mu \right\} t_*^{-m-1}; \tag{10}$$

$$|J(t)''| \leq (a+1)\mu |J(t)'|^{(\mu-1)/\mu} |\psi|_\infty, \tag{11}$$

where (7) *is not necessary for the lower bound in* (8).

Proof. We have

$$J(t) = \int_t^\infty r^{-m-1} \left\{ r^{-a+b+p-1} \int_0^r s^a \psi \right\}^\mu \leq \int_t^\infty r^{-m-1} \left(\int_0^\infty s^{b+p-1} \psi \right)^\mu$$

on one hand and

$$J(t) \leq \int_0^1 \left(\int_0^r \psi \right)^\mu + \int_1^\infty r^{-m-1} \left(\int_0^\infty s^{b+p-1} \psi \right)^\mu$$

on the other hand; the RHS of (8) then follows from integrations by parts . For $t \leq 1$,

$$J(t) \geq \int_1^\infty \left(r^{-a} \int_0^r s^a \psi \right)^\mu \geq \left(\int_0^1 s^a \psi \right)^\mu \int_0^\infty r^{-a\mu} dr$$

and for $t > 1$,

$$J(t) \geq \left(\int_0^1 s^a \psi \right)^\mu \int_t^\infty r^{-a\mu} dr.$$

We thus get the LHS of (8).
If $b = a + 1 - p$, $J(t) \geq (\int_0^1 s^a\psi)^\mu \int_t^\infty r^{-m-1}dr$ and (9) follows.
For $t > 1$, as $a > b + p - 1$,

$$0 \leq -J(t)' \leq \left(t^{-b+1-p}\int_0^t s^{b+p-1}\psi\right)^\mu \leq t^{-m-1}\left(\int_0^\infty s^{b+p-1}\psi\right)^\mu.$$

For $t \leq 1$ $|J(t)'| \leq (\int_0^1 \psi)^\mu$ and (10) is obtained.
For (11),

$$J(r)'' = -\mu\left\{r^{-a}\int_0^r s^a\psi\right\}^{\mu-1}\left\{-ar^{-a-1}\int_0^r s^a\psi(s) + \psi(r)\right\}$$

hence from

$$|J(r)''| \leq \mu(a+1)|\psi|_\infty\left(r^{-a}\int_0^\infty s^a\psi\right)^{\mu-1}$$

(11) follows.

Lemma 4. *Under the assumptions* (6)–(7)

$$(r^a|U'|^{p-2}U')' + r^a\psi(r) = 0; \quad r \geq 0 \tag{12}$$

has a decreasing and positive solution $U \in C^2([0,\infty))$ *such that*

$$U(r) \leq (1+r)^{-b/(p-1)} \qquad \forall r \geq 0. \tag{13}$$

Proof. It is easy to verify that $U = J$ where J is defined in (5) satisfies (12). Then (8)–(11) complete the proof.

2.2 Proof of Theorem 1

Let u and v be two such solutions with $u > v > 0$ in some $[0, R)$.
As they are decreasing, from the equations, in $[0, R)$

$$\{r^a(|v'|^{p-1} - |u'|^{p-1})\}' = r^a\{F_q^\nu(r,v) - F_q^\nu(r,u)\} > 0$$

with $r^a(|v'|^{p-1} - |u'|^{p-1})|_{r=0} = 0$, whence $|v'| > |u'|$ or $v' < u' \leq 0$ in $(0, R)$.
This implies that $u(r) - v(r) > u(0) - v(0)$ whenever $v(r) > 0$.

2.3 Proof of Theorem 2

In the lights of the super-sub-solutions methods established in [3], it suffices for each case to find an appropriate sub- or supersolution of the problem.
1) The function U in Lemma 4 is a supersolution of (2) as

$$\psi(r) = f(r)(1+r)^{b\gamma/(p-1)} \leq f(r)U(r)^{-\gamma}.$$

The estimate for the case $b = a + 1 - p$ follows from (9).
2) i) The solution v, say, obtained in 1) satisfies $v(r) \le (1+r)^{-b/(p-1)}$.

$$F(r,v) = v^q\{f(r)v^{-(\gamma+q)} - \nu\} \ge v^q\{f(r)(1+r)^{b(\gamma+q)/(p-1)} - \nu\}.$$

So, as $f(r) > 0$ everywhere, there is $\nu_0 := \inf_{r>0}[f(r)(1+r)^{b(q+\gamma)/(p-1)}]$ such that if $\nu \le \nu_0$, then $F(r,v) := f(r)v^{-\gamma} - \nu v^q \ge 0$ and $\partial_v F(r,v) \le 0$.
Then v is a suitable subsolution of (3) as the condition (1) of Theorem 5 of [3] is guaranteed by $q > \max\{p(p-1)/b, -\gamma + \theta(p-1)/b$ (see (ϕ)).
If in addition $q > (b+p)(p-1)/b$, then $V(r) := \int_r^\infty (\int_0^t (s/t)^a F(s,v)ds)^\mu dt$ is a supersolution of the equation with $r^{b/(p-1)}V(r)$ bounded.
ii) For $G(r,\phi) := \nu f(r)\phi^{-\gamma} + \phi^q$,

$$\partial_\phi G(r,\phi) = q\phi^{-1-\gamma}\{\phi^{q+\gamma} - \gamma\nu f(r)/q\} := q\Phi^{-1-\gamma}\Psi_\nu(r),$$

where $\Psi_\nu(r) := (1+r)^{-b(\gamma+q)/(p-1)} - \nu\gamma f(r)/q$.
If $q > \theta(p-1)/b - \gamma$ and $\phi < (1+r)^{-b/(p-1)}$, then for some large $R > 0$ there is $\Psi_\nu(r) < 0$; in this case there is $\nu_1 := \sup_{[0,R]} q\{\gamma(1+r)^{b(\gamma+q)/(p-1)} f(r)\}$ such that $\nu > \nu_1$ implies that G is decreasing in such positive ϕ. The solution v obtained in 1) is then a suitable supersolution of (4).

This work is dedicated to my late uncle Toam Chatue J.B., († on 14/08/1997).

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