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Rothe's Method for Degenerate Quasilinear Parabolic Equations

Volker Pluschke

Department of Mathematics and Computer Science,
Martin-Luther-University Halle-Wittenberg, P.O.Box,
06 099 Halle, Germany
Email: pluschke@mathematik.uni-halle.de
WWW: http://www.mathematik.uni-halle.de/~analysis/pluschke

Abstract. In the contribution we state local existence of a weak solution u to a degenerate quasilinear parabolic Dirichlet problem. Degeneration occurs in the coefficient $g(x, t, u) \geq 0$ in front of the time derivative, which is not assumed to be bounded below and above, resp., by positive constants. The nonlinear coefficients and the right-hand side are defined with respect to u only in a neighbourhood of the initial function.
The quasilinear parabolic problem is approximated by linear elliptic problems by means of semidiscretization in time (Rothe's method). We obtain L_∞-estimates for the approximations and uniform convergence to a Hölder continuous weak solution. An essential tool for this are estimates of the first order derivatives uniformly for all subdivisions in the space $L_\infty([0, T], L_\nu(G))$ with certain $\nu > 2$.

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1 Introduction

In this contribution we formulate a local existence result for the parabolic initial boundary value problem

$$g(x, t, u)\, u_t + A(t, u)u = f(x, t, u) \qquad \text{in } Q_T, \tag{1}$$

$$u(x, t) = 0 \qquad \text{on } \Gamma, \tag{2}$$

$$u(x, 0) = U_0(x) \qquad x \in G, \tag{3}$$

where

$$A(t, v)u = -\sum_{i,k=1}^{N} \frac{\partial}{\partial x_k}\left(a_{ik}(x, t, v)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} a_i(x, t, v)\frac{\partial u}{\partial x_i}, \tag{4}$$

The paper is an overview article summarizing the results of [8] and [9].

which we obtain by means of semidiscretization in time (Rothe's method). Here we denote by $G \subset \mathbb{R}^N, N \geq 2$, a simply connected, bounded domain with boundary $\partial G \in C^1,\ I = [0, T],\ Q_T = G \times I,\ \Gamma = \partial G \times I$.

In the following we give an overview on the results of the author's papers [8] and [9]. In [8] the non-degenerated quasilinear problem (1)–(3) is investigated where $g \equiv 1$. The aim of the paper [9] is to deal with the case where the coefficient $g(\cdot, t, u)$ may degenerate (i.e. $g = 0$ or $g = \infty$) on some sets $S_{t,u} \subset G$ with $\mathrm{meas}(S_{t,u}) = 0$ (there $A = A(t)$ is linear).

In both papers it is supposed that the nonlinear coefficients and the right-hand side f are defined only in a neighbourhood

$$\mathcal{M}_R(U_0) = \{(x, t, u) \in \mathbb{R}^{N+2}\ :\ x \in G,\ t \in I,\ |u - U_0(x)| \leq R\}$$

of the initial function for some given $R > 0$. In order to have convergence of the Rothe approximations to a solution we have to ensure that the approximations belong to the ball

$$\mathcal{B}_R(U_0) = \{u \in C(\bar{G}) : \|u - U_0\|_{C(\bar{G})} \leq R\}.$$

This holds for small time $t \in \hat{I} = [0, \hat{T}]$ due to L_∞-estimates which are derived by means of the technique of Moser [7] combined with recursive estimates due to Alikakos [1]. Because of the nonlinear coefficient g we only can apply this technique to the semidiscrete problem if we have uniform boundedness of the discrete time derivative δu_j in $L_\infty(I, L_\nu(G))$ with sufficiently large $\nu > 2$. In standard literature on Rothe's method (cf. Kačur [2], Chapter 2) such an estimate of δu_j is derived under the assumption of monotonicity of the nonlinear operator A. Moreover, one obtains this estimate for $\nu = 2$ only. Without assumption of full monotonicity and with nonlinear coefficients in general one only obtains an estimate in $L_2(I, L_2(G))$ (cf. e.g. Kačur [4], Lemma 2.7, where a similar problem with nonlinear coefficient of u_t is treated). We use L_p-theory with $p > 2$, power-type test functions, interpolation arguments, and nonlinear Gronwall lemma to derive the desired a priori estimate. Moreover, degeneration forces to work in weighted Lebesgue spaces.

These strong a priori estimates also yield strong convergence results for the approximates despite of weak regularity of the data (Lebesgue data). We obtain uniform convergence of the Rothe functions in Hölder space with respect to space and time variables.

2 Preliminaries and result

We use standard notations of the function and evolution spaces, resp. (cf. [5]). By $\|\cdot\|_p$, $\|\cdot\|_{1,p}$, and $\|\cdot\|_{0,\lambda}$ we denote the norms in $L_p(G)$, $W_0^{1,p}(G)$, and $C^\lambda(\bar{G})$, respectively. $L_{p,g}(G)$ denotes the weighted Lebesgue space with norm $\|u\|_{p,g} = (\int_G g|u|^p\, dx)^{1/p}$ for nonnegative $g \in L_1(G)$. $\langle\cdot,\cdot\rangle$ is the duality between $L_p(G)$ and $L_{p'}(G)$ $(1/p + 1/p' = 1)$. For $t \in I$ and $v \in C(\bar{G})$ the operator $A(t, v)$ from (4) generates a bilinear form on $W_0^{1,p}(G) \times W_0^{1,p'}(G)$ denoted by $A_{(t,v)}(\cdot,\cdot)$.

First we formulate the complete assumptions which we fix throughout the paper.

Assumptions. For given $R > 0$ let g, a_{ik} a_i, and f be Carathéodory functions defined on $\mathcal{M}_R(U_0)$. Let further r_1, r_2, r_3, μ_1, μ_2, μ_3, μ_4, ν_1, ν, σ, κ be real numbers fulfilling the relations $2 \le \kappa < \infty$, $r_1 > N$, $r_2 > \frac{N(\kappa-1)}{2\kappa-N}$, $r_3 > \frac{N}{2}$, $\frac{N}{2} < \mu_1 \le \nu_1 = \frac{\sigma}{\sigma+1}\kappa$, $\mu_i \le \nu < \frac{N\kappa}{N-2}$ $(i = 2,3,4)$, $\frac{N\kappa}{\kappa-2} < \mu_2$, $\frac{N\kappa}{2\kappa-2} < \mu_3$, $\frac{N\kappa}{2\kappa+N-2} < \mu_4$, $\sigma > 1$.

Then we suppose for arbitrary $t, t' \in I$ and $u, u' \in \mathcal{B}_R(U_0)$

(i) $U_0 \in \overset{o}{W}{}^1_{r_1}(G)$, $A(0,U_0)U_0 \in L_1(G)$;

(ii) $g(\cdot,t,u) : I \times \mathcal{B}_R(U_0) \to L_{r_2}(G)$ is bounded in $L_{r_2}(G)$ and fulfils the Lipschitz condition

$$\|g(\cdot,t,u) - g(\cdot,t',u')\|_{\mu_1} \le l_1\,(|t-t'| + \|u-u'\|_{\nu_1}).$$

Furthermore, $g(x,t,u) \ge 0$ for all $(x,t,u) \in \mathcal{M}_R(U_0)$ and $1/g(\cdot,t,u) : I \times \mathcal{B}_R(U_0) \to L_\sigma(G)$ is bounded in $L_\sigma(G)$.

(iii) $a_{ik}(\cdot,t,u) : I \times \mathcal{B}_R(U_0) \to C(\bar G)$ and $a_i(\cdot,t,u) : I \times \mathcal{B}_R(U_0) \to L_\infty(G)$ are bounded mappings which fulfil the Lipschitz conditions

$$\|a_{ik}(\cdot,t,u) - a_{ik}(\cdot,t',u')\|_{\mu_2} \le l_2\,(|t-t'| + \|u-u'\|_\nu)$$
$$\|a_i(\cdot,t,u) - a_i(\cdot,t',u')\|_{\mu_3} \le l_3\,(|t-t'| + \|u-u'\|_\nu)$$

as well as ellipticity condition $(a > 0)$

$$\textstyle\sum_{i,k} a_{ik}(x,t,v)\,\xi_i\xi_k \ge a\,\xi^2 \qquad \text{for all } (x,t,v) \in \mathcal{M}_R(U_0) \text{ and } \xi \in \mathbb{R}^N.$$

(iv) $f(\cdot,t,u) : I \times \mathcal{B}_R(U_0) \to L_{r_3}(G)$ is bounded in $L_{r_3}(G)$ and fulfils the Lipschitz condition

$$\|f(\cdot,t,u) - f(\cdot,t',u')\|_{\mu_4} \le l_4\,(|t-t'| + \|u-u'\|_\nu).$$

(v) It holds the compatibility condition

$$\big(f(\cdot,0,U_0) - A(0,U_0)U_0\big)/g(\cdot,0,U_0) \in L_{\kappa,g(\cdot,0,U_0)}(G)\ .$$

We remark that the coefficient g may not only decay to zero on some sets (this decay is governed by the assumption $1/g \in L_\infty(I, L_\sigma(G))$ but it also may have singularities because it belongs to the Lebegue space $L_\infty(I, L_{r_2}(G))$. This is equivalent to some degeneration of ellipticity of the operator A.

In order to solve the problem by semidiscretization in time (Rothe's method) we subdivide the time interval I by points $t_j = jh$ $(h > 0, j = 0,\dots,n)$ and replace (1)–(3) by the time discretized problem (in weak formulation)

$$\langle g_j\,\delta u_j, v\rangle + A_j(u_j,v) = \langle f_j, v\rangle \qquad \forall v \in \overset{o}{W}{}^1_{p'}(G)\ , \tag{1_j}$$

$$u_j = 0 \qquad \text{on } \partial G\ , \tag{2_j}$$

$$u_0 = U_0\ , \tag{3_0}$$

$j = 1, \dots, n$, where $\delta u_j := (u_j - u_{j-1})/h$, $g_j := g(x, t_j, u_{j-1})$, $f_j := f(x, t_j, u_{j-1})$, and $A_j(\cdot,\cdot) := A_{(t_j, u_{j-1})}(\cdot,\cdot)$. This is a set of linear elliptic boundary value problems to determine the approximate u_j if u_{j-1} is already known. However, we do not know if $u_j \in \mathcal{B}_R(U_0)$, i.e. whether the data of (1_{j+1}), (2_{j+1}) are well-defined because of the local assumptions. Therefore we define global extensions

$$\psi^R(x,t,u) = \begin{cases} \psi(x,t,u) & \text{for } (x,t,u) \in \mathcal{M}_R(U_0) \\ \psi(x,t,U_0(x) + R \operatorname{sign}(u - U_0(x)) & \text{otherwise} \end{cases}$$

and replace g, a_{ik}, a_i, and f by g^R, a_{ik}^R, a_i^R, and f^R, respectively. By Lemma 3 we state that $u_j \in \mathcal{B}_R(U_0)$ for $t_j \le \hat{T}$, hence we omit the superscript R.

For sufficiently small fixed h now we can solve these elliptic boundary value problems applying the Lax-Milgram theorem (after an interpolation procedure to deal with the weighted norm with weight g_j) and a regularity theorem (cf. [6, Theorem 5.5.4']).

Lemma 1. *Let assumptions (i)–(iv) be fulfilled. Then there are $h_0 > 0$, $r > N$ such that for $0 < h \le h_0$ the problem (1_j), (2_j), (3_0) has a unique solution $u_j \in \overset{o}{W}{}^1_r(G)$, $j = 1, \dots, n$.*

Especially, the embedding theorem implies continuity of u_j.

By interpolation with respect to time we obtain the Rothe functions

$$\tilde{u}^n(x,t) = \frac{t_j - t}{h} u_{j-1}(x) + \frac{t - t_{j-1}}{h} u_j(x) \quad , \quad t \in [t_{j-1}, t_j]$$

and

$$\bar{u}^n(x,t) = \begin{cases} u_j(x) & \text{if } t \in (t_{j-1}, t_j], \\ U_0(x) & \text{if } t \le 0. \end{cases}$$

Our result is the following

Theorem 2. *Suppose assumptions (i)–(v). Then the following assertions hold:*

a) *There is an interval $\hat{I} = [0, \hat{T}]$ such that problem (1)–(3) has a unique weak solution u with $u(\cdot,t) \in \mathcal{B}_R(U_0)$ for any $t \in \hat{I}$ fulfilling for all $v \in L_1(\hat{I}, \overset{o}{W}{}^1_{r'}(G) \cap L_{\varrho'}(G))$ $(\varrho' = r_2' \kappa')$ the relation*

$$\int_{\hat{I}} \langle g(\cdot,t,u) u_t \,,\, v \rangle \, dt + \int_{\hat{I}} A_{(t,u)}(u \,,\, v) \, dt = \int_{\hat{I}} \langle f(\cdot,t,u) \,,\, v \rangle \, dt$$

and initial condition (3).

b) *The solution u belongs to the spaces*

$$u \in C^\alpha(\bar{Q}_{\hat{T}}) \cap L_\infty(\hat{I}, \overset{o}{W}{}^1_r(G)) \cap W^1_\infty(\hat{I}, L_{\nu_1}(G))$$

for some $r > N$ and $\alpha > 0$. Moreover, $u_t \in L_\kappa(\hat{I}, L_s(G))$ for $s < \frac{N\kappa}{N-2}$ and $g(\cdot,\cdot,u)u_t \in L_\infty(\hat{I}, L_\varrho(G))$ with $\varrho = \frac{r_2 \kappa}{r_2 + \kappa - 1}$.

c) *The solution u is pointwise uniformly approximated by the Rothe functions $\tilde{u}^n$ with further convergence properties*

$$\begin{aligned} \tilde{u}^n &\longrightarrow u && \text{in } C^\alpha(\bar{Q}_{\hat{T}}) \\ \tilde{u}^n, \bar{u}^n &\longrightarrow u && \text{in } L_\infty(\hat{I}, C^\lambda(\bar{G})) \qquad (\lambda < 1 - N/r) \\ \tilde{u}^n, \bar{u}^n &\overset{*}{\rightharpoonup} u && \text{in } L_\infty(\hat{I}, \overset{o}{W}{}^1_r(G)) \\ \tilde{u}^n_t &\overset{*}{\rightharpoonup} u_t && \text{in } L_\infty(\hat{I}, L_{\nu_1}(G)) \end{aligned}$$

as n tends to infinity.

d) *It holds an error estimate*

$$\sup_{t \in \hat{I}} \|\tilde{u}^n(\cdot, t) - u(\cdot, t)\|_{\nu_1} \le c\, h_n^{1/2} .$$

The $r > N$ may be explicitly given in terms of the Lebesgue exponents from the assumptions. Furthermore, because of uniform boundedness of the approximations in $L_\infty(\hat{I}, \overset{o}{W}{}^1_r(G))$ an interpolation inequality yields an error estimate in Hölder space, too,

$$\sup_{t \in \hat{I}} \|\tilde{u}^n(\cdot, t) - u(\cdot, t)\|_{0,\lambda} \le c\, h_n^{(1-\lambda-N/r)/2} , \quad 0 < \lambda < 1 - N/r .$$

3 A priori estimates for the approximations

In this final section we sketch some steps of the proof of Theorem 2. For the details compare [8] and [9].

In order to prove $u_j \in \mathcal{B}_R(U_0)$ we have to estimate $z_j := u_j - U_0$ in $L_\infty(G)$. Therefore, we rewrite (1_j) into

$$\langle g_j\, \delta z_j, v \rangle + A_j(z_j, v) = \langle f_j, v \rangle - A_j(U_0, v) \qquad \forall v \in \overset{o}{W}{}^1_{r'}(G). \tag{5}$$

We use the Moser iteration technique [7]. The idea of this technique is to estimate $\|z\|_p$ for arbitrary $p \ge p_0$ and then pass with p to infinity. Since $\|z\|_p \to \|z\|_\infty$ (cf. [2, Theorem 2.11.4]) one obtains an estimate in $L_\infty(G)$. In our case, because of degeneration, we have to work with the weighted norm $\|z\|_{p,g}$. But we have the same property $\|z\|_{p,g} \to \|z\|_\infty$ as $p \to \infty$, i.e. the influence of the degeneration vanishes in the limit. In order to obtain an estimate of the weighted $L_{p,g}$-norm for arbitrary p we test (5) with $v = |z_j|^{p-2} z_j$ and obtain after some manipulations

$$\begin{aligned} &\|z_j\|^p_{p,g_{j+1}} - \|z_{j-1}\|^p_{p,g_j} + ch\, \|w_j\|^2_{1,2} \\ &\quad \le ch\, \|z_j\|^p_p + cph\, \|f_j\|_{r_3} \|z_j\|^{p-1}_{r_3'(p-1)} + cph\, \|U_0\|_{1,r_0} \|w_j\|_{1,2} \|z_j\|^{(p-2)/2}_s \\ &\qquad + cph\, (1 + \|\delta u_j\|_{\nu_1})\, \|z_j\|^p_{\mu_1' p} , \end{aligned} \tag{6}$$

where $w_j := |z_j|^{(p-2)/2} z_j$. The last item on the right-hand side appears since the item $\|z_j\|^p_{p,g_j}$ arising from (5) on the left-hand side must be replaced by $\|z_j\|^p_{p,g_{j+1}}$. Hence, for our L_∞-estimate we need uniform boundedness of $\|\delta u_j\|_{\nu_1}$ (cf. Lemma 3). Then we have to estimate the unweighted norms of z_j on the right-hand side by weighted norms occuring on the left-hand side. For this we use the Nirenberg-Gagliardo interpolation inequality

$$\|w\|_s \le C\,\|w\|^{\theta}_{1,2}\,\|w\|^{1-\theta}_1$$

for some $\theta \in (0,1)$, $s < 2N/(N-2)$. This enables us to insert the weight by means of Cauchy-Schwarz' inequality

$$\|w_j\|_1 = \|z_j\|^{p/2}_{p/2} \le \|1/g_{j+1}\|^{1/2}_1\,\|z_j\|^{p/2}_{p,g_{j+1}}. \tag{7}$$

Summing up the inequalities (6) for $j = 1,\dots,i$ we would then come to an estimate of the weighted norm $\|z_i\|_{p,g_{i+1}}$. However, in our case it is not possible to hold the bounds depending on p uniformly bounded as $p \to \infty$. Therefore, for the limit process we use a recursive approach due to Alikakos [1]. Since $1/g \in L_\sigma(G)$ with $\sigma > 1$ is supposed we may even obtain the weighted norm $\|z_j\|_{\lambda p,g_{j+1}}$ with $\lambda < 1$ on the right-hand side of (7). Then we derive an recursive estimate of the form

$$\max_{t_j\le t}\|z_j\|^p_{p,g_{j+1}} \;\le\; cp^c\,t\left(\max_{t_j\le t}\|z_j\|^p_{\lambda p,g_{j+1}} + \max_{t_j\le t}\|z_j\|^{\beta(p)p}_{\lambda p,g_{j+1}}\right)$$

which is investigated for the special sequence $p_k = \lambda^{-k}p_0$. Passing to the limit $k \to \infty$ this yields an estimate of $\|z_j\|_\infty$ in terms of $\|z_j\|_{p_0,g_{j+1}}$ for fixed p_0. After estimation of this norm for fixed p_0 we obtain

Lemma 3. *Let be $\|\delta u_j\|_{\nu_1} \le C$ for $j = 1,\dots,n$ independent of the subdivision. Then there are constants $c,\gamma > 0$ such that*

$$\max_{0\le t_j\le t}\|u_j - u_0\|_\infty \le c\,t^\gamma\,.$$

Obviously, since $u_j \in C(\bar{G})$ due to Lemma 1 we have $u_j \in \mathcal{B}_R(U_0)$ for all $t_j \in \hat{I} := [0,\hat{T}]$ if we fix $\hat{T}$ for given $R > 0$ by $c\hat{T}^\gamma = R$.

Remark 4. In [3, Theorem 4.17] J. Kačur proves a L_∞-estimate for quasilinear equations without the assumption $\|\delta u_j\| \le C$ of our Lemma 3. The reason is that the degeneration in [3] corresponds to the case $g(x,t,s) = b'(s)$, i.e. the item concerning the time derivative is written in the form $b(u)_t$. This is not possible in our case. Moreover, we have weaker regularity of the data with respect to x. On the other hand, the technique in [3] allows stronger degeneration with respect to u.

The next task is to check the assumption of Lemma 3. One obtains an estimate of δu_j by forming the difference (1_j)–(1_{j-1}) and testing the resulting relation with an appropriate test function (cf. [2, Chapters 2.1, 2.2]). However, we run into problems since we have no full monotonicity of the nonlinear operator $Au := A(t,u)u$ unlike in [2, Example 2.2.17]. Testing (1_j)–(1_{j-1}) with $v = |\delta u_j|^{\kappa-2}\delta u_j$ in order to estimate the weighted norm $\|\delta u_j\|_{\kappa,g_j}$ we are forced to deal with an item

$$ch\,(1 + \|\delta u_{j-1}\|_{\nu})\,\|u_j\|_{1,r}\|\omega_j\|_{1,2}\|\delta u_j\|_s^{(\kappa-2)/2} \tag{8}$$

$(\omega_j = |\delta u_j|^{(\kappa-2)/2}\delta u_j)$ arising from $\big(A_j - A_{j-1}\big)(u_j, v)$ on the right-hand side. Hence, we have to estimate the space-like derivative $\|u_j\|_{1,r}$ in order to obtain an estimate of the discrete time derivative. This is possible by means of a priori estimates for elliptic equations like (1_j) is. However, we are not able to split $g_j\,\delta u_j$ into $g_j\frac{u_j}{h}$ and $g_j\frac{u_{j-1}}{h}$,resp., and then to use estimates of the solution u_j of the elliptic equation with right-hand side $f_j + g_j\frac{u_{j-1}}{h}$ since we need a priori bounds uniformly with respect to $h > 0$. Hence we write (1_j) in the form

$$A_j u_j = f_j - g_j\,\delta u_j =: F_j$$

where we obtain from L_p-theory for elliptic equations (cf. [6, Theorem 5.5.5'] the estimate

$$\|u_j\|_{1,r} \le c\,(\|F_j\|_\rho + \|u_j\|_1) \le c\left(1 + \sum_{i=1}^{j}\|\delta u_i\|_{\nu_1}\right). \tag{9}$$

The constant c now is independent of the subdivision since the coefficients of the elliptic operator A_j are uniformly bounded. Inserting this estimate into (8) we notice that the total power of $\|\delta u_j\|$ on the right-hand side of the resulting estimate is $\kappa + 1$ while we have the power κ on the left-hand side, only. This seems to contradict the intention to obtain boundedness of $\|\delta u_j\|$ by these estimations. However, after some very technical manipulations, we are able to apply a nonlinear discrete version of the Gronwall lemma (cf. Willett, Wong [10, Theorem 4]) to obtain at least a local bound for small t_j. Since the unweighted norm $\|\delta u_j\|_{\nu_1}$ may be estimated by the weighted norm $\|\delta u_j\|_{\kappa,g_j}$ we obtain

Lemma 5. *Suppose assumptions* (i)–(v). *Then for $h \le h_0$ there is a time interval $[0,T^*]$ such that the estimate*

$$\|\delta u_j\|_{\nu_1} \;\le\; C_1 \qquad \forall t_j \in [0,T^*]$$

holds independent of the subdivision.

By means of (9) this lemma also yields boundedness of the space-like derivatives.

Lemma 6. *For all $h \le h_0$ the estimate*

$$\|u_j\|_{1,r} \;\le\; C_2 \qquad \forall t_j \in [0,T^*]$$

holds independent of the subdivision.

The time T^* is a bound for the length of our local existence interval $\hat{I}$. If $\hat{T} > T^*$ for the $\hat{T}$ choosen after Lemma 3 we have to fix $\hat{I} := [0,T^*]$.

The a priori estimates from Lemma 3, 5, and 6 now provide the tools to prove the convergence results of Theorem 2.

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