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The Existence of Global Solutions to the Elliptic-Hyperbolic Davey-Stewartson System with Small Initial Data

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Abstract. We study the initial value problem for the elliptic-hyperbolic Davey-Stewartson system. Our purpose in this paper is to prove global existence of small solutions of this system in the usual weighted Sobolev space $H^{3,0} \cap H^{0,3}$. Furthermore we prove L^∞ time decay estimates in L^∞ of solutions u such that $\|u(t)\|_{L^\infty} \le C(1+|t|)^{-1}$.

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We study the initial value problem for the Davey-Stewartson systems

$$\begin{cases} i\partial_t u + c_0\partial_{x_1}^2 u + \partial_{x_2}^2 u = c_1|u|^2 u + c_2 u\partial_{x_1}\varphi, & (x,t) \in \mathbf{R}^3, \\ \partial_{x_1}^2\varphi + c_3\partial_{x_2}^2\varphi = \partial_{x_1}|u|^2, \\ u(x,0) = \phi(x), \end{cases} \tag{1}$$

where $c_0, c_3 \in \mathbf{R}$, $c_1, c_2 \in \mathbf{C}$, u is a complex valued function and φ is a real valued function. The systems (1) for $c_3 > 0$ were derived by Davey and Stewartson [4] and model the evolution equation of two-dimensional long waves over finite depth liquid. Djordjevic-Redekopp [5] showed that the parameter c_3 can become negative when capillary effects are significant. When (c_0, c_1, c_2, c_3) $= (1, -1, 2, -1)$, $(-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system (1) is referred as the DSI, DSII defocusing and DSII focusing respectively in the inverse scattering literature. In [7], Ghidaglia and Saut classified (1) as elliptic-elliptic, elliptic- hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of (c_0, c_3) : $(+,+), (+,-), (-,+)$ and $(-,-)$. For the elliptic-elliptic and hyperbolic-elliptic cases, local and global properties of solutions were studied in [7] in the usual Sobolev spaces L^2, H^1 and H^2. In this paper we consider

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the elliptic-hyperbolic case. In this case after a rotation in the x_1x_2-plane and rescaling, the system (1) can be written as

$$\begin{cases} i\partial_t u + \Delta u = d_1|u|^2 u + d_2 u\partial_{x_1}\varphi + d_3 u\partial_{x_2}\varphi, \\ \partial_{x_1}\partial_{x_2}\varphi = d_4\partial_{x_1}|u|^2 + d_5\partial_{x_2}|u|^2, \end{cases} \tag{2}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $d_1, \cdots, d_5$ are arbitrary constants. In order to solve the system of equations, one has to assume that $\varphi(\cdot)$ satisfies the radiation condition, namely, we assume that for given functions φ_1 and φ_2

$$\lim_{x_2\to\infty}\varphi(x_1,x_2,t) = \varphi_1(x_1,t) \quad \text{and} \quad \lim_{x_1\to\infty}\varphi(x_1,x_2,t) = \varphi_2(x_2,t). \tag{3}$$

Under the radiation condition (3), the system (2) can be written as

$$\begin{aligned} i\partial_t u + \Delta u = d_1|u|^2 u &+ d_2 u\int_{x_2}^{\infty}\partial_{x_1}|u|^2(x_1,{x_2}',t)d{x_2}' \\ &+ d_3 u\int_{x_1}^{\infty}\partial_{x_2}|u|^2({x_1}',x_2,t)d{x_1}' + d_4 u\partial_{x_1}\varphi_1 + d_5 u\partial_{x_2}\varphi_2 \end{aligned} \tag{4}$$

with the initial condition $u(x,0) = \phi(x)$. In what follows we consider the equation (4).

In this paper we use the following notations.

Notations. We define the weighted Sobolev space as follows

$$\begin{aligned} H^{m,l} &= \{f \in L^2; \|(1-\partial_{x_1}^2-\partial_{x_2}^2)^{m/2}(1+{x_1}^2+{x_2}^2)^{l/2}f\| < \infty\}, \\ H^{m,l}(\mathbf{R}_{x_j}) &= \{f \in L^2(\mathbf{R}_{x_j}); \|(1-\partial_{x_j}^2)^{m/2}(1+{x_j}^2)^{l/2}f\|_{L^2(\mathbf{R}_j)} < \infty\}, \end{aligned}$$

where $\|\cdot\|$ denotes the usual L^2 norm. We let $\partial = (\partial_{x_1}, \partial_{x_2})$, $J = (J_{x_1}, J_{x_2})$, $J_{x_j} = x_j + 2it\partial_{x_j}$. For simplicity we write $L^p_{x_j} = L^p(\mathbf{R}_{x_j})$, $L^p_{x_1}L^q_{x_2} = L^p(\mathbf{R}_{x_1}; L^q(\mathbf{R}_{x_2}))$, $H^{m,l}_{x_j} = H^{m,l}(\mathbf{R}_{x_j})$, $\|\cdot\|_{X^{m,l}(t)} = \sum_{|\alpha|\le m}\|\partial^\alpha\cdot\| + \sum_{|\alpha|\le l}\|J^\alpha\cdot\|$, where $\alpha = (\alpha_1,\alpha_2)$, $|\alpha| = \alpha_1+\alpha_2$, $\alpha_1,\alpha_2 \in \mathbf{N}\cup\{0\}$.

Local existence of small solutions to (4) was shown when the initial function is in $H^{m,l}$ in [13] for $H^{12,0}\cap H^{0,6}$, [8] for $H^{m,0}\cap H^{0,l}$, $(m,l>1)$, [1] for $H^{m,0}$, (m is sufficiently large integer) and [9] for $H^{m,0}$, $(m \ge 5/2)$. Furthermore in [11] without smallness condition on the data local existence of solutions was proved in the analytic function space which consists of real analytic functions. Global existence of small solutions to (4) was also given in [11] when the data are real analytic and satisfy the exponential decay condition.

Recently, H. Chihara [1] established the global existence of small solutions to (4) in higher order Sobolev spaces. Our purpose in this paper is to prove the global existence of small solutions to (4) in the usual weighted Sobolev spaces $H^{3,0}\cap H^{0,3}$, which is considered as lower order Sobolev class compared to one used in [1], by the calculus of commutator of operators. We shall prove

Theorem 1. *Let* $\phi \in H^{3,0} \cap H^{0,3}$, $\partial_{x_1}^{j+1}\varphi_1 \in C(\mathbf{R}; L^\infty_{x_1})$, $\partial_{x_2}^{j+1}\varphi_2 \in C(\mathbf{R}; L^\infty_{x_2})$, $(0 \le j \le 3)$, ε_3 *and* δ_3 *be sufficiently small, where*

$$\varepsilon_m = \sup_{t\in\mathbf{R}} \sum_{0\le j\le m} (1+t)^{1+a}\Big(\|(t\partial_{x_1})^j \partial_{x_1}\varphi_1(t)\|_{L^\infty_{x_1}} + \|\partial_{x_1}^{j+1}\varphi_1(t)\|_{L^\infty_{x_1}} + \|(t\partial_{x_2})^j \partial_{x_2}\varphi_2(t)\|_{L^\infty_{x_2}} + \|\partial_{x_2}^{j+1}\varphi_2(t)\|_{L^\infty_{x_2}}\Big), \quad a>0,$$

$$\delta_m \ge \Big(\sum_{|\alpha|+|\beta|\le m} \|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} {x_1}^{\beta_1} {x_2}^{\beta_2}\phi\|^2 \Big)^{1/2}.$$

Then there exists a unique global solution u *of* (4) *such that*

$$u \in L^\infty_{local}(\mathbf{R}; H^{3,0}\cap H^{0,3}) \cap C(\mathbf{R}; H^{2,0}\cap H^{0,2}), \tag{5}$$

$$\sup_{t\in\mathbf{R}} \Big(\sum_{|\alpha|+|\beta|\le 2} \|\partial^\alpha J^\beta u(t)\| + \sum_{|\alpha|+|\beta|\le 3} (1+t)^{-C\delta_3} \|\partial^\alpha J^\beta u(t)\| \Big) \le 4\delta_3. \tag{6}$$

Corollary 2. *Let* u *be the solution constructed in Theorem 1. Then we have*

$$\|u(t)\|_{L^\infty} \le C(1+|t|)^{-1}(\|\phi\|_{H^{3,0}} + \|\phi\|_{H^{0,3}}).$$

Moreover, for any $\phi \in H^{3,0}\cap H^{0,3}$ *there exist* $u^\pm$ *such that*

$$\|u(t) - U(t)u^\pm\|_{H^{2,0}} \to 0 \quad \text{as} \quad t \to \pm\infty,$$

where $U(t) = e^{it(\partial_{x_1}^2 + \partial_{x_2}^2)}$.

The rate of decay obtained in Corollary 2 is the same as that of solutions to linear Schrödinger equations. Time decay of solutions for the Davey-Stewartson systems (1) was obtained in [3,7] when $(c_0, c_3) = (+,+)$ and $(c_0, c_3) = (-,+)$ and in [11] when $(c_0, c_3) = (+,-)$ and $(c_0, c_3) = (-,-)$ under exponential decay conditions on the data.

We try to explain our strategy of the proof of Theorem 1. For simplicity we consider following equation

$$i\partial_t u + \Delta u = u \int_{x_2}^\infty \partial_{x_1}|u|^2 dx_2{}',$$

which have only main nonlinear term. We use following operator K_{x_1} and K_{x_2}, where

$$K_{x_1} = K_{x_1}(v) = \sum_{m=0}^\infty \frac{A^m}{m!}\Big(\int_{-\infty}^{x_1} \|v(t, x_1{}')\|^2_{L^2_{x_2}} dx_1{}' \frac{D_{x_1}}{\langle D_{x_1}\rangle}\Big)^m$$

and

$$K_{x_2} = K_{x_2}(v) = \sum_{m=0}^\infty \frac{A^m}{m!}\Big(\int_{-\infty}^{x_2} \|v(t, x_2{}')\|^2_{L^2_{x_1}} dx_2{}' \frac{D_{x_2}}{\langle D_{x_2}\rangle}\Big)^m.$$

So, if we take $A^2 = 1/\delta_3$ (for the definition of δ_3, see Theorem 1), by virtue of commutator estimates and following lemma,

Lemma 3.

$$\|[\langle D_{x_1}\rangle^{1/2}, f]g\|_{L^2_{x_1}} + \|[\langle D_{x_1}\rangle, f]g\|_{L^2_{x_1}} \le C\|\langle D_{x_1}\rangle f\|_{L^\infty_{x_1}}\|g\|_{L^2_{x_1}},$$

which follows from Coifman and Meyer's result (see [2, p. 154]), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2)\\
&\quad + \frac{1}{4\delta_3^{1/2}}\sum_{|\alpha|+|\beta|\le 3}\Big(\big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}K_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\big\|^2_{L^2_{x_1}}\\
&\quad + \big\|\|u(t)\|_{L^2_{x_1}}\|\langle D_{x_2}\rangle^{1/2}K_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\big\|^2_{L^2_{x_2}}\Big)\\
&\quad \le C(1+A)^2(1+t)^{-1}\|u(t)\|^2_{X^{2,2}(t)}(1+\|u(t)\|^2_{X^{2,2}(t)})\|u(t)\|^2_{X^{3,3}(t)}\\
&\quad + \sum_{|\alpha|+|\beta|\le 3}\Bigg(\left|\mathrm{Im}(K_{x_1}\partial^\alpha J^\beta u\int_{x_2}^\infty \partial_{x_1}|u|^2 dx_2{}', K_{x_1}\partial^\alpha J^\beta u)\right|\\
&\qquad\qquad + \left|\mathrm{Im}(K_{x_2}\partial^\alpha J^\beta u\int_{x_2}^\infty \partial_{x_1}|u|^2 dx_2{}', K_{x_2}\partial^\alpha J^\beta u)\right|\Bigg).
\end{aligned}\tag{7}$$

The second term of the left hand side of (7) means smoothing properties of solutions to the equation. So we have to estimate the term

$$\begin{aligned}
\sum_{|\alpha|+|\beta|\le 3}\Bigg(&\left|\mathrm{Im}(K_{x_1}\partial^\alpha J^\beta u\int_{x_2}^\infty \partial_{x_1}|u|^2 dx_2{}', K_{x_1}\partial^\alpha J^\beta u)\right|\\
&+ \left|\mathrm{Im}(K_{x_2}\partial^\alpha J^\beta u\int_{x_2}^\infty \partial_{x_1}|u|^2 dx_2{}', K_{x_2}\partial^\alpha J^\beta u)\right|\Bigg).
\end{aligned}$$

For this purpose, we pay attention to the special structure of the nonlinear term

$$u\int_{x_2}^\infty \partial_{x_1}|u|^2 dx_2{}' = u\frac{1}{2it}\int_{x_2}^\infty \bar{u}J_{x_1}u - u\overline{J_{x_1}u}\,dx_2{}'.\tag{8}$$

This deformation shows this nonlinear term has own time decay in some sence. Using this structure, we can estimate as following,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2)\\
&+ \left(\frac{1}{4\delta_3^{1/2}} - Ce^{C\delta_3}\right)\sum_{|\alpha|+|\beta|\le 3}\Big(\big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}K_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\big\|^2_{L^2_{x_1}}\\
&+ \big\|\|u(t)\|_{L^2_{x_1}}\|\langle D_{x_2}\rangle^{1/2}K_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\big\|^2_{L^2_{x_2}}\Big) \le C(1+t)^{-1}\delta_3\|u(t)\|^2_{X^{3,3}(t)}
\end{aligned}\tag{9}$$

provided that δ_3 is sufficiently small and

$$\sup_{-T\le t\le T} \|u(t)\|^2_{X^{2,2}(t)} \le 4\delta_3^2, \tag{10}$$

$$\sup_{-T\le t\le T} (1+|t|)^{-C\delta_3}\|u(t)\|^2_{X^{3,3}(t)} \le 4\delta_3^2 \tag{11}$$

for some time $T > 0$. We choose δ_3 satisfying

$$\frac{1}{4\delta_3^{1/2}} - Ce^{C\delta_3} \ge 0.$$

Then we have

$$\|u(t)\|^2_{X^{3,3}(t)} \le e^{C\delta_3}\delta_3^2 + C\delta_3 \int_0^t (1+s)^{-1}\|u(s)\|^2_{X^{3,3}(t)}ds. \tag{12}$$

Thus (9) shows that the nonliear term is controlled by the second term of the left hand side of (7) and the right hand side of (9). Global existence theorem is obtained by showing that (10) and (11) hold for any T. In order to prove (10) and (11) for any $T > 0$ we need (12) and the following inequality

$$\|u(t)\|^2_{X^{2,2}(t)} \le e^{C\delta_3}\delta_3^2 + C\delta_3 \int_0^t (1+s)^{-1-2C\delta_3}\|u(s)\|^2_{X^{3,3}(t)}ds. \tag{13}$$

The inequality (13) is obtained by the structure of nonlinear term (8) again.

Theorem 1 is obtained by applying the Gronwall inequality to (12) and (13). It seems to be difficult to get the inequality (12) through the methods used in [8,9], because nonlinear terms are not taken into account to derive smoothing properties of solutions in [8,9]. On the other hand the operators K_{x_1} and K_{x_2} are made based on the nonlocal nonlinear terms (the second and the third terms on the right hand side of (4)). The similar operators as those of K_{x_1} and K_{x_2} have been used in [1] to obtain Theorem 0.1 and the local existence theorem of small solutions to (4) in the usual order Sobolev space.

Remark 4. We cannot apply above method to hyperbolic-hyerbolic Davey-Stewartson system. In fact, if we estimate similarly as above, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2)\\
&\quad + \frac{1}{4\delta_3^{1/2}}\sum_{|\alpha|+|\beta|\le 3}(\big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}\tilde{K}_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\big\|^2_{L^2_{x_1}}\\
&\quad + \big\|\|u(t)\|_{L^2_{x_1}}\|\langle D_{x_2}\rangle^{1/2}\tilde{K}_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\big\|^2_{L^2_{x_2}})\\
&\quad \le C(1+A)^2(1+t)^{-1}\|u(t)\|^2_{X^{2,2}(t)}(1+\|u(t)\|^2_{X^{2,2}(t)})\|u(t)\|^2_{X^{3,3}(t)}\\
&\qquad + Ce^{C\delta_3}\sum_{|\alpha|+|\beta|\le 3}\big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}\tilde{K}_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\big\|^2_{L^2_{x_1}}
\end{aligned} \tag{14}$$

under the condition (10) and (11), where

$$\tilde{K}_{x_1} = \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\int_{-\infty}^{x_2} \|v(t, x_2{}')\|_{L^2_{x_1}}^2 dx_2{}' \frac{D_{x_1}}{\langle D_{x_1} \rangle} \right)^m = e^{A \int_{-\infty}^{x_2} \|v(t,x_2{}')\|_{L^2_{x_1}}^2 dx_2{}' \frac{D_{x_1}}{\langle D_{x_1} \rangle}}$$

and

$$\tilde{K}_{x_2} = \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\int_{-\infty}^{x_1} \|v(t, x_1')\|_{L^2_{x_2}}^2 dx_1{}' \frac{D_{x_2}}{\langle D_{x_2} \rangle} \right)^m = e^{A \int_{-\infty}^{x_1} \|v(t,x_1')\|_{L^2_{x_2}}^2 dx_1{}' \frac{D_{x_2}}{\langle D_{x_2} \rangle}},$$

but we can easy to see that the last term of the right-hand side of (14) cannot controlled by the second term of the left-hand side of (14).

Remark 5. For Davey-Stewartson systems, we can define formally the energy similar as the usual nonlinear Schrödinger equation if $c_1, c_2 \in \mathbf{R}$. But unfortunately, this energy is not conserved in elliptic-hyperbolic and hyperbolic-hyperbolic cases. So, we cannot use the usual H^1 a-priori estimate by energy. This is one reason that the global existence theorem of this system is difficult.

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