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Some Examples for the Extended Use of the Parametric Representation Method

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Abstract. The Parametric Representation Method had been applied successfully to construct bifurcation diagrams relating to equilibria of dynamical systems whenever the equilibria are determined from a single equation containing two control parameters linearly. The Discriminant-curve (that is the saddle-node bifurcation curve parametrized by the state variable remained after the elimination) is the base of this method, as it had been shown. The number and even the value of the stationary state variables can be derived from that.
Here we show some possible extensions of the method via two examples.
1. Nonlinear parameter dependence
2. Reaction-diffusion equations,
Similarly to the above simple case, the PRM provides us with information about the stationary solutions. Although some features do not remain valid for these extensions.

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1 Introduction

The parametric representation method is a geometric tool for the study of stationary solutions of differential equations depending on two parameters. There are well-known methods [12,20,22] giving the bifurcation parameter values (where the number or stability of stationary solutions changes), and serving with information on the stationary solutions if the parameters are in a small neighbourhood of the bifurcation values.

Our aim is to divide the whole parameter space according to the number and the type of the stationary points. We shall call this separation the global bifurcation diagram;'global' refers here to the parameter space, while our investigation is local in the phase space. The first theoretical result in this direction was achieved by Rabinowitz [16]. He followed the changes of one stationary state

This is the final form of the paper.

varying a single parameter value. Details and other references can be found in [5]. In [3,4,9] there are methods for constructing global bifurcation diagrams which give the number of roots of polynomials. In many practical applications the construction of such bifurcation diagrams was carried out by ad hoc methods [11,13,15].

The Parametric Representation Method (PRM) [9] is a systematic approach, which is especially useful if the parameter dependence of the system is simpler than the dependence on the state variables. As an example, in chemical dynamical systems the parameter dependence is usually linear, therefore the PRM is easy to apply [2,14,17]. Some general features of the method together with a pictorial algorithm for determination of the exact number of stationary points can be found in [6,7]. PRM was also applied to study the root structure of polynomials and extended to study their complex roots [8]. This method is also a useful tool to reveal some relations between the saddle-node and Hopf bifurcation diagrams [17,19].

We summarize the main results concerning the case of linear parameter dependence in Section 2, the detailed study can be found in [18]. In Section 3 it is shown on an example how can be used the PRM, when the equation (determining the stationary states) contains parameters non-linearly. In Section 4 we illustrate that the PRM may be useful for the determination of stationary solutions of a scalar reaction-diffusion equation.

2 Linear parameter dependence

We want to give the number of the stationary points of the following ODE:

$$\dot{x}(t) = F(x(t), u),$$

where $F : R^n \times R^k \to R^n$ is a differentiable function, $x(t) \in R^n$ is the vector of state variables and $u \in R^k$ is the vector of parameters. The first step before executing the global bifurcation analysis is the reduction of the dimension of the system. There is no general method for that, the optimal one depends on the structure of the concrete system. The Liapunov-Schmidt reduction or — for polynomials — the Euclidean algorithm are often useful tools. In this section we assume that

— the system of algebraic equations $F(x, u) = 0$ giving the stationary points is already reduced to a single equation and

— we have two control parameters, u_1 and u_2, which are involved in the right hand side of the reduced equation linearly. These control parameters may also be functions of the original parameters of the system. (Two control parameters are chosen regularly in practical applications, primarily because of the visualization.)

With these assumptions the above general problem reduces to the following one:

Problem. Let us divide the parameter plane (u_1, u_2) with respect to the number of the solutions of equation

$$f(x, u_1, u_2) := f_0(x) + f_1(x)u_1 + f_2(x)u_2 = 0, \tag{1}$$

where $f_i \in C^2$ and $f_1^2(x) + f_2^2(x) \neq 0$ for all $x \in R$; and give a geometric method determining the number and values of the solutions at a given parameter pair (u_1, u_2).

According to the implicit function theorem the number of solutions may change when the parameter values cross the *singularity set*:

$$S = \{(u_1, u_2) \in R^2 : \exists\, x \in R \ , f(x, u_1, u_2) = f'(x, u_1, u_2) = 0\},$$

where prime denotes differentiation with respect to x. The detailed study of singularities can be found in [1,10]. The PRM has the following advantages: 1. the singularity set can be easily constructed as a curve parametrized by x, called D-curve; 2. the solutions belonging to a given parameter pair can be determined by a simple geometric algorithm based on the tangential property; 3. the global bifurcation diagram, that divides the parameter plane according to the number of solutions can be geometrically constructed with the aid of the D-curve.

Now let us see how to apply the PRM for (1). Concerning the singularity set the determinant

$$\Delta(x) := f_1(x) f_2'(x) - f_1'(x) f_2(x)$$

plays a crucial role. For simplicity we assume that $\Delta(x) \neq 0$ for all $x \in R$ (the general case, when Δ may have zeros is considered in [18]). Then the system

$$f_0(x) + f_1(x)u_1 + f_2(x)u_2 = 0, \tag{2}$$
$$f_0'(x) + f_1'(x)u_1 + f_2'(x)u_2 = 0, \tag{3}$$

has one and only one solution for (u_1, u_2). These equations determine the D-curve for this case:

Definition. The solution of the system (2), (3) for u_1 and u_2 is called **D-curve** (or discriminant curve). The point belonging to x will be denoted by $D(x) = (D_1(x), D_2(x))$, i.e.

$$u_1 = \frac{f_2(x) f_0'(x) - f_2'(x) f_0(x)}{\Delta(x)} =: D_1(x),$$
$$u_2 = \frac{f_1(x) f_0'(x) - f_1'(x) f_0(x)}{\Delta(x)} =: D_2(x).$$

Thus we produced the singularity set as a curve parametrized by x.

Let us introduce the straight lines:

$$M(x) := \{(u_1, u_2) \in R^2 : f(x, u_1, u_2) = 0\},$$

i.e. the set of parameter pairs for which a given number x is a solution of (1). The main point of the PRM is the fact that the D-curve (the singularity set) is the envelope of these lines. This fact is involved in the following theorem, which will be referred to as **tangential property**.

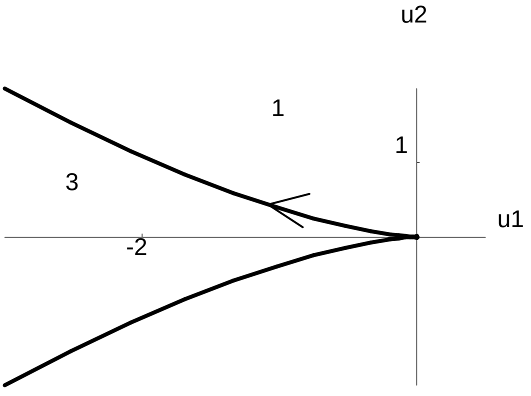

Fig. 1.

Theorem 1. *The line $M(x)$ is tangential to the D-curve at the point $D(x)$.*

For the proof see [18] Theorem 4.

Corollary 1. *The number of solutions of (1) belonging to a given parameter pair (u_1, u_2) is equal to the number of tangents, which can be drawn to the D-curve from the point (u_1, u_2).*

Thus as a solution of our problem we got the following:
Geometric algorithm. Draw the D-curve belonging to our equation. Given a parameter pair (u_1, u_2) any tangent from this point to the D-curve gives a solution x of the equation; the value of x can be read on the D-curve at the tangential point.

As an illustration let us consider the equation

$$x^3 + u_1 x + u_2 = 0.$$

The D-curve is determined by the system

$$x^3 + u_1 x + u_2 = 0,$$
$$3x^2 + u_1 = 0.$$

From this system we get

$$D_1(x) = -3x^2, \qquad D_2(x) = 2x^3.$$

The D-curve is depicted in Fig. 1. If (u_1, u_2) is on the left side of the D-curve, then the equation has three solutions, because we can draw three tangents from (u_1, u_2) to the D-curve. If (u_1, u_2) is on the right side of the D-curve, then there is one solution, because we can draw one tangent from (u_1, u_2). The value of x on the D-curve is increasing with increasing u_2 and it is zero at the origin.

The determination of the number of the tangents is facilitated by the so-called **convexity property**: the D-curve consists of convex arcs that join together in cusp points. To be more formal we cite Theorem 5 from [18]:

Theorem 2. *Suppose that the roots of the function*

$$B(x) := f_0''(x) + f_1''(x)D_1(x) + f_2''(x)D_2(x)$$

are isolated.

(i) *If B changes its sign at x_0 then the D-curve has a cusp point at x_0.*

(ii) *If B does not change its sign at x_0 then the D-curve is locally on the left (right) side of its tangent belonging to x_0 if $\Delta(x_0)$ is positive (negative).*

The D-curve also gives the global bifurcation diagram (GBD) i.e. the curve (or system of curves) which divides the parameter plane into regions within which the number of solutions of (1) is constant. The construction of the GBD is based on the fact that the number of roots of a function may change in two ways:

1. it has a multiple root (the derivative vanishes at a root),
2. a root goes to (or comes from) the infinity.

The GBD consists of the D-curve and its tangents or asymptotes (if they exist) at the points belonging to $x \to \infty$ and $x \to -\infty$. For the exact formulation see Theorem 6 in [18].

3 Nonlinear parameter dependence

In this section we apply the PRM for the special equation

$$x^2 + u_1^2 x + u_2 = 0. \tag{4}$$

Our aim is to divide the parameter plane (u_1, u_2) according to the number of solutions ($x \in R$) of (4). The singularity set is determined by (4) and

$$2x + u_1^2 = 0. \tag{5}$$

The solution of the system (4)–(5) is

$$u_1^2 = -2x, \qquad u_2 = x^2. \tag{6}$$

Thus the singularity set can not be parametrized by x, but we can define the D-curve with two branches D^+ and D^- as follows (see Fig. 2.):

Definition. The two solutions of (4)–(5) for (u_1, u_2) are called the two branches of the D-curve for $x \le 0$, i.e.

$$D_1^+(x) = \sqrt{-2x}, \qquad D_2^+(x) = x^2,$$
$$D_1^-(x) = -\sqrt{-2x}, \qquad D_2^-(x) = x^2.$$

Similarly as in Section 2 let us introduce

$$M(x) := \{(u_1, u_2) \in R^2 : x^2 + u_1^2 x + u_2 = 0\},$$

which is in this case a parabola for a given $x \in R$. The singularity set is the envelope of the parabolas belonging to $x < 0$, therefore the tangential property holds in the following form:

Theorem 3. *For a fixed $x < 0$ the parabola $M(x)$ is tangential to the D^+ and D^- curves at the points $D^+(x)$ and $D^-(x)$.*

The tangential property does not refer to the values $x > 0$, however, it is obvious from (6) that there is no singularity for these values. Therefore the parabolas $M(x)$ belonging to $x > 0$ do not intersect each other, they form a one-fold cover of the lower half plane. Thus the number of solutions can be given by the following:

Geometric algorithm. Given a parameter pair (u_1, u_2) any tangential parabola of the form (4) from this point to the D-curve gives a solution x of the equation (4); the value of x can be read on the D-curve at the tangential point (the value of x is the same on the D^+ and on the D^- branches). If the parameter pair is in the upper half plane, then the number of the tangential parabolas is equal to the number of solutions. If the parameter pair is in the lower half plane, then the number of solutions is more by one than the number of tangential parabolas.

Using this algorithm we get that the number of solutions of (4) is 0 if (u_1, u_2) is above the D-curve, and it is 2 if (u_1, u_2) is below the D-curve, see Fig. 2.

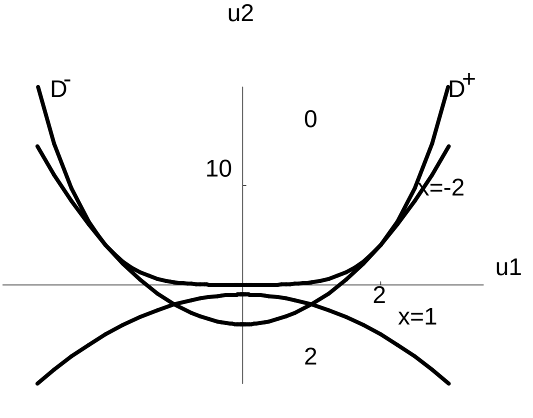

Fig. 2.

4 Application of the PRM to a reaction-diffusion equation

Let us consider the reaction-diffusion equation

$$\partial_t u(t,x) = \partial_{xx} u(t,x) + f(u(t,x),a,b)$$

with the boundary condition

$$u(t,0) = u(t,1) = 0,$$

where $f : R^3 \to R$ is a differentiable function. The stationary solutions are determined by the following boundary value problem:

$$v''(x) + f(v(x),a,b) = 0, \tag{7}$$

$$v(0) = v(1) = 0. \tag{8}$$

We will study the following:
Problem. Divide the parameter plane (a,b) according to the number of the solutions of (7)–(8).

The boundary value problem (7)–(8) is usually [20,21] reduced to the phase plane analysis of the system

$$v' = w, \tag{9}$$

$$w' = -f(v,a,b). \tag{10}$$

If we have a $p \in R$, such that the trajectory $t \to (v(t),w(t))$ of (9)–(10) starting from $(0,p)$ reaches the vertical axis $(v = 0)$ at time 1 (i.e. $v(1) = 0$), then v is a solution of (7)–(8). Therefore the time map T is defined that measures the time an orbit takes to get from the point $(0,p)$ to the vertical axis. This time is the double of that one the orbit takes to get from the point $(0,p)$ to the horizontal axis (say, at point $(0,q)$), because the flow of (9)–(10) is symmetric to the horizontal axis. System (9)–(10) has the first integral

$$H(v,w) = \frac{w^2}{2} + F(v),$$

where $F(v) = \int_0^v f(s)\,ds$. This first integral enables us to calculate the time map explicitly:

$$T(p) = 2\int_0^q \frac{1}{\sqrt{2(F(q)-F(v))}}\,dv.$$

The relation between p and q is given by the first integral: $F(q) = \frac{p^2}{2}$. Thus the time map can be regarded as a function of q, a and b:

$$S(q,a,b) = 2\int_0^q \frac{1}{\sqrt{2(F(q)-F(v))}}\,dv.$$

A solution q of

$$S(q,a,b)=1 \tag{11}$$

gives a solution of (7)–(8). However, several solutions of (7)–(8) may give the same q as a solution of (11). Therefore our problem partially reduces to the following one:

Problem. Divide the parameter plane according to the number of solutions of (11).

This problem is similar to that one dealt with in Section 3 (the parameter dependence may be more complicated). Using the PRM we can define the D-curve (singularity set) belonging to equation (11). It is determined by the equations:

$$S(q,D_1(q),D_2(q))=1,$$
$$\partial_q S(q,D_1(q),D_2(q))=0.$$

Solving these equations numerically one can get a curve on the parameter plane (a,b), which divides it into regions according to the number of solutions of (11).

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