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Multiple Solutions of Nonlinear Boundary Value Problems and Topological Degree

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Abstract. This paper deals with the second order nonlinear boundary value problems. We consider the two-point, multipoint or nonlinear boundary conditions on a compact interval and suppose the existence of strict upper and lower solutions of the problem with the both types of ordering i.e. the lower (upper) solution is less than the upper (lower) one. We prove the relation between the topological degree and strict upper and lower solutions in the both cases and using this we get the existence and multiplicity results for the boundary value problems under consideration.

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1 Introduction

When we study the boundary value problems for the second order differential equation

$$x'' = f(t, x, x'), \tag{1.1}$$

with certain linear or nonlinear boundary conditions on the compact interval $J = [a, b] \subset \mathbf{R}$ we often use the properties of lower and upper solutions for (1.1). Let us remind the definition.

Let f be continuous on $J \times \mathbf{R}^2$ (or let f satisfy the Carathéodory conditions on $J \times \mathbf{R}^2$). The functions $\sigma_1, \sigma_2 \in C^2(J)$ (or $AC^1(J)$) are called lower and upper solutions for (1.1), if they satisfy

$$\begin{aligned} \sigma_1''(t) &\geq f(t, \sigma_1(t), \sigma_1'(t)), \\ \sigma_2''(t) &\leq f(t, \sigma_2(t), \sigma_2'(t)), \end{aligned} \tag{1.2}$$

for all $t \in J$ (for a.e. $t \in J$). If the inequalities in (1.2) are strict, then σ_1, σ_2 are called strict lower and upper solutions.

We distinguish two basic cases:

1. The functions σ_1, σ_2 are well ordered, i.e.

$$\sigma_1(t) \le \sigma_2(t) \text{ for all } t \in J. \tag{1.3}$$

2. The functions σ_1, σ_2 are not well ordered, i.e. the condition (1.3) falls.

The most existence results concern the first case, but there are the existence results for the second case, as well. We can refer to the papers [7], [3] or [4].

Here, we want to present the existence and multiplicity results for (1.1) (with various boundary conditions) in the first case and also in the second case where σ_1, σ_2 have the opposite order, i.e.

$$\sigma_2(t) \le \sigma_1(t) \text{ for all } t \in J. \tag{1.4}$$

Our results are based on the relation between the topological degree of the operator corresponding to the boundary value problem and strict lower and upper solutions fulfilling (1.3) or (1.4) (in the strict sense).

For getting the existence and multiplicity results we need a priori estimates of solutions of the original boundary value problem or of solutions of proper auxiliary boundary value problems. Working with σ_1, σ_2, we want to estimate the solutions just by σ_1, σ_2. For the estimation at the endpoints a, b of J we use certain connection between σ_1, σ_2 and the boundary conditions. It is well known that for the classical two-point boundary conditions such connection has the form:

– for the periodic conditions

$$x(a) = x(b), \ x'(a) = x'(b), \tag{1.5}$$

we suppose

$$\sigma_i(a) = \sigma_i(b), \ \big(\sigma_i'(b) - \sigma_i'(a)\big)(-1)^i \ge 0, \ i = 1, 2; \tag{1.6}$$

– for the Neumann conditions

$$x'(a) = 0, \ x'(b) = 0, \tag{1.7}$$

we assume

$$\sigma_i'(a)(-1)^i \le 0, \ \sigma_i'(b)(-1)^i \ge 0, \ i = 1, 2. \tag{1.8}$$

Similarly,

– for the four-point conditions

$$x(a) = x(c), \ x(d) = x(b), \ a < c \le d < b, \tag{1.9}$$

σ_1, σ_2 have to satisfy

$$\begin{gathered} \big(\sigma_i(c) - \sigma_i(a)\big)(-1)^i \le 0, \\ \big(\sigma_i(b) - \sigma_i(d)\big)(-1)^i \ge 0, \ i = 1, 2, \end{gathered} \tag{1.10}$$

– for the nonlinear conditions

$$g_1\big(x(a), x'(a)\big) = 0,\ \ g_2\big(x(b), x'(b)\big) = 0, \tag{1.11}$$

where $g_1, g_2 \in C(\mathbf{R}^2)$ are increasing in the second argument and g_1 is nonincreasing and g_2 nondecreasing in the first argument, we can impose on σ_1, σ_2

$$\begin{aligned} &g_1\big(\sigma_i(a), \sigma_i'(a)\big)(-1)^i \le 0, \\ &g_2\big(\sigma_i(b), \sigma_i'(b)\big)(-1)^i \ge 0,\ \ i = 1, 2. \end{aligned} \tag{1.12}$$

Let us note that for more general nonlinear two-point boundary conditions the compatibility of the boundary conditions with σ_1, σ_2 was introduced in [14]. For the special cases of the conditions (1.5), (1.7) and (1.11) this notion leads just to the assumptions (1.6), (1.8) and (1.12).

In this paper we will study the boundary value problems (1.1), (k), and we will assume the existence of lower and upper solutions σ_1, σ_2 of (1.1) with the property (k+.1), k∈{1.5,1.7,1.9,1.11}. The problem (1.1), (k), k∈{1.5,1.7,1.9,1.11}, can be written in the form of the operator equation

$$(L + N)\,x = 0, \tag{1.13}$$

where $L : \operatorname{dom} L \to Y$ is a linear operator and it is a Fredholm map of index 0, and $N : C^1(J) \to Y$ is, in general, nonlinear and it is L-compact on any open bounded set $\Omega \subset C^1(J)$. The form of L and N and the choice of the spaces $\operatorname{dom} L$ and Y depend on the type of boundary value problems. Let us suppose that f is continuous on $J \times \mathbf{R}^2$. Then we put for k∈{1.5,1.7,1.9} $\operatorname{dom} L = \{x \in C^2(J) : x$ satisfies (k)$\}$, $Y = C(J)$, $L : x \longmapsto x''$, $N : x \longmapsto -f(\cdot, x(\cdot), x'(\cdot))$; for the boundary condition (1.11) we put $\operatorname{dom} L = C^2(J)$, $Y = C(J) \times \mathbf{R}^2$, $L : x \longmapsto (x'', 0, 0)$, $N : x \longmapsto \big(-f(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b))\big)$. For more details see [2], [8], [9].

If the equation (1.13) has no solution on the boundary of Ω then there exists the degree of the map $L + N$ in Ω with respect to L

$$d_L(L + N, \Omega).$$

In [6], the relation between the degree and strict lower and upper solutions satisfying (1.3) (in the strict sense) is shown. In the following section we will formulate this relation for the above boundary value problems.

2 Topological degree for f bounded

First, let us suppose that $f \in C(J \times \mathbf{R}^2)$ is bounded:

$$\exists M \in (0, \infty) : |f(t, x, y)| < M \text{ for } \forall (t, x, y) \in J \times \mathbf{R}^2. \tag{2.1}$$

For f unbounded we will use the method of a priori estimates and replace the condition (2.1) by the conditions of the growth or sign types in the next sections.

Theorem 1. *Suppose* k $\in\{$1.5,1.7,1.9,1.11$\}$. *Let* (2.1) *be fulfilled,* (1.13) *be the operator equation corresponding to the problem*(1.1), (k) *and let* σ_1, σ_2 *be strict lower and upper solutions of* (1.1), (k) *with*

$$\sigma_1(t) < \sigma_2(t) \text{ for all } t \in J.$$

Then

$$d_L(L+N, \Omega_1) = 1 \pmod 2, \tag{2.2}$$

with

$$\begin{aligned}\Omega_1 =& \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t), |x'(t)| < c \text{ for all } t \in J\},\\ & \text{where } c \geq (2M + r + 1)(b-a) \text{ for k} \in \{1.5, 1.7, 1.9\}\\ & \text{and } c \geq (2M + r + 1)(b-a) + 2(r+1)/(b-a) \text{ for k}=1.11,\\ & r = \| \sigma_1 \|_{\max} + \| \sigma_2 \|_{\max} .\end{aligned}$$

Theorem 1 concerns the case of well ordered σ_1, σ_2. the case where σ_1, σ_2 are ordered by the opposite way is described in Theorem 2.

Theorem 2. *Suppose* k $\in\{$1.5,1.7,1.9,1.11$\}$. *Let* (2.1) *be fulfilled,* (1.13) *be the operator equation corresponding to the problem* (1.1), (k) *and let* σ_1, σ_2 *be strict lower and upper solutions of* (1.1), (k) *satisfying*

$$\sigma_2(t) < \sigma_1(t) \text{ for all } t \in J.$$

Then

$$d_L(L+N, \Omega_2) = 1 \pmod 2, \tag{2.3}$$

where

$$\begin{aligned}\Omega_2 = \{x \in C^1(J) : \|x\|_{\max} < A,\ \|x'\|_{\max} < B,\\ \exists t_x \in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\},\end{aligned}$$

with $B \geq 2(b-a)M$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + 2(b-a)^2 M$ *for* k $\in\{$1.5,1.7,1.9$\}$, $B \geq 2(b-a)M + \|\sigma_2'\|_{\max}$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ *for* k=1.11.

Corollary 3. *Suppose* k $\in\{$1.5,1.7,1.9,1.11$\}$. *If* σ_1, σ_2 *in Theorem* 1 (2) *are not strict, then either the problem* (1.1), (k) *has a solution on* $\partial\Omega_1$ $(\partial\Omega_2)$ *or the condition* (2.2) ((2.3)) *is valid.*

3 Existence and multiplicity for f bounded

As the direct consequence of Corollary 3, using a limiting process, we obtain the following existence results for the problems (1.1), (k), k $\in\{$1.5,1.7,1.9,1.11$\}$.

Theorem 4. *Suppose* k ∈{1.5,1.7,1.9,1.11}*. Let* (2.1) *be fulfilled and let* σ_1, σ_2 *be lower and upper solutions of* (1.1), (k) *with*

$$\sigma_1(t) \le \sigma_2(t) \text{ for all } t \in J.$$

Then the problem (1.1), (k) *has at least one solution in* $\overline{\Omega}_1$*, where* Ω_1 *is the set from Theorem* 1*.*

Remark 5. The assumption about the monotonicity of g_1, g_2 can be omitted in Theorem 1 and 4. The existence results of Theorem 4 are known and they are presented here for the completeness, only.

Theorem 6. *Suppose* k ∈{1.5,1.7,1.9,1.11}*. Let* (2.1) *be fulfilled and let* σ_1, σ_2 *be lower and upper solutions of* (1.1), (k) *with*

$$\sigma_2(t) \le \sigma_1(t) \text{ for all } t \in J.$$

Then the problem (1.1), (k) *has at least one solution in* $\overline{\Omega}_2$*, where* Ω_2 *is the set from Theorem* 2*.*

Remark 7. For k ∈{1.5,1.7} the similar existence results are proven in [3], [4], [7].

Theorems 1 and 2 are a tool for proving multiplicity results for (1.1), (k), both for the linear two-point (k ∈{1.5,1.7}) or multipoint boundary conditions (k=1.9) and for the nonlinear boundary condition (k=1.11).

Theorem 8. *Suppose* k ∈{1.5,1.7,1.9,1.11}*. Let* (2.1) *be fulfilled and let* σ_1*,* σ_2*,* σ_3 *be strict lower, upper and lower solutions of* (1.1), (k) *with*

$$\sigma_1(t) < \sigma_2(t) < \sigma_3(t) \text{ for all } t \in J. \tag{3.1}$$

Then (1.1), (k) *has at least two different solutions* u*,* v *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t), \sigma_1(t) < v(t) \text{ for all } t \in J,$$
$$\sigma_2(t_v) < v(t_v) < \sigma_3(t_v) \text{ for a } t_v \in J.$$

The dual situation is described in Theorem 9.

Theorem 9. *Let all assumptions of Theorem* 8 *be fulfilled with the exception that now* $\sigma_1, \sigma_2, \sigma_3$ *are strict lower, upper and upper solutions with*

$$\sigma_3(t) < \sigma_1(t) < \sigma_2(t) \text{ for all } t \in J \tag{3.2}$$

Then (1.1), (k) *has at least two different solutions* u*,* v *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t), v(t) < \sigma_2(t) \text{ for all } t \in J,$$
$$\sigma_3(t_v) < v(t_v) < \sigma_1(t_v) \text{ for a } t_v \in J.$$

For constant lower and upper solutions we get the multiplicity result of the Ambrosetti-Prodi type.

Theorem 10. *Suppose* k$\in${1.5,1.7,1.9}. *Let* (2.1) *be fulfilled and let* $n \in \mathbf{N}$, $n \geq 2$, $s_1, r_1, \dots, r_{n+1} \in \mathbf{R}$ *be such that*

$$r_1 < r_2 < \cdots < r_{n+1} \tag{3.3}$$

and

$$\big(f(t, r_i, 0) - s_1\big)(-1)^i < 0 \text{ for all } t \in J,\ i \in \{1, \dots, n\}. \tag{3.4}$$

Then there exist $s_2, s_3 \in (-M, s_1)$, $s_3 \leq s_2$, *such that the problem*

$$x'' + f(t, x, x') = s,\ (k) \tag{3.5}$$

has:

(i) *at least* n *different solutions greater than* r_1 *for* $s \in (s_2, s_1]$;
(ii) *at least* $\frac{n+1}{2}$ $(\frac{n}{2})$ *solutions greater than* r_1 *for* $s = s_2$ *and* n *odd (even);*
(iii) *provided* $s_3 < s_2$ *at least one solution greater then* r_1 *for* $s \in [s_3, s_2)$;
(iv) *no solution for* $s < s_3$.

4 Topological degree for f unbounded

In this section we suppose that k$\in$ {1.5, 1.7, 1.11}, that (1.13) is the operator equation corresponding to the problem (1.1), (k) and that σ_1, σ_2 are strict lower and upper solutions of (1.1), (k).

Using the method of a priori estimates we can replace the condition (2.1) in Theorem 1 by the Nagumo-Knobloch-Schmitt condition with bounding functions φ_1, φ_2 :

$$\begin{gathered}\exists \varphi_1, \varphi_2 \in C^1(K) : \varphi_1(t, \sigma_i(t)) \leq \sigma_i'(t), \varphi_2(t, \sigma_i(t)) \geq \sigma_i'(t),\\ f(t, x, \varphi_1(t, x) < \tfrac{\partial \varphi_1(t,x)}{\partial t} + \tfrac{\partial \varphi_1(t,x)}{\partial x}\, \varphi_1(t, x),\\ f(t, x, \varphi_2(t, x) > \tfrac{\partial \varphi_2(t,x)}{\partial t} + \tfrac{\partial \varphi_2(t,x)}{\partial x}\, \varphi_2(t, x),\\ \text{for } i \in \{1, 2\} \text{ and for all } (t, x) \in K = J \times [\sigma_1(t), \sigma_2(t)].\end{gathered} \tag{4.1}$$

Theorem 11. *Let* (4.1) *be fulfilled and let*

$$\sigma_1(t) < \sigma_2(t) \text{ for all } t \in J.$$

Further suppose that for k=1.5

$$(\varphi_i(b, x) - \varphi_i(a, x))(-1)^i \geq 0,$$

for k=1.7

$$(\varphi_i(b, x) - \sigma_i'(b))(-1)^i > 0,$$

and for k=1.11

$$g_2(x, \varphi_i(b,x))(-1)^i > 0,$$

with $i = 1, 2,\ x \in [\sigma_1(t), \sigma_2(t)]$.
Then

$$d_L(L+N, \Omega_3) = 1 \pmod 2,$$

where

$$\Omega_3 = \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t),\ \ \varphi_1(t,x) < x'(t) < \varphi_2(t,x) \ \text{on}\ K\}.$$

For the constant functions $\sigma_1, \sigma_2, \varphi_1, \varphi_2$ Theorem 11 implies

Corollary 12. *Suppose that there exist real numbers* $r_1 < r_2$, $c_1 < 0 < c_2$, *such that*

$$f(t, r_1, 0) < 0,\ f(t, r_2, 0) > 0, \tag{4.2}$$
$$f(t, x, c_1) < 0,\ f(t, x, c_2) > 0, \tag{4.3}$$

for all $(t, x) \in J \times [r_1, r_2]$.
If k=1.11 *we suppose moreover that for* $x \in [r_1, r_2]$

$$\begin{aligned} g_1(r_1, 0) \ge 0,\ \ g_1(r_2, 0) \le 0, \\ g_2(r_1, 0) \le 0,\ \ g_2(r_2, 0) \ge 0, \end{aligned} \tag{4.4}$$
$$g_2(x, c_i)(-1)^i > 0,\ \ i = 1, 2. \tag{4.5}$$

Then

$$d_L(L+N, \Omega_4) = 1 \pmod 2,$$

where

$$\Omega_4 = \{x \in C^1(J) : r_1 < x(t) < r_2,\ c_1 < x'(t) < c_2, \forall\, t \in J.\}$$

Now, let us consider the special case of bounding functions depending on t only:

$$\begin{aligned} &\exists \beta_1, \beta_2 \in C^1(J)\text{: } \beta_1(t) \le \sigma_i'(t),\ \beta_2(t) \ge \sigma_i'(t), \\ &f(t, x, \beta_1(t)) < \beta_1'(t),\ f(t, x, \beta_2(t)) > \beta_2'(t), \end{aligned} \tag{4.6}$$

for all $(t, x) \in J \times [s_2, s_1]$, where $s_2 = \min\{\sigma_2(t) : t \in J\} - \int_a^b \gamma(t)dt$, $s_1 = \max\{\sigma_1(t) : t \in J\} + \int_a^b \gamma(t)dt$, $\gamma(t) = \max\{|\beta_1(t)|, |\beta_2(t)|\}$.

Theorem 13. *Let* (4.6) *be fulfilled and let*

$$\sigma_2(t) < \sigma_1(t) \text{ for all } t \in J.$$

Further suppose that for k=1.5

$$(\beta_i(b) - \beta_i(a))(-1)^i \geq 0, \tag{4.7}$$

for k=1.7

$$(\beta_i(b) - \sigma_i'(b))(-1)^i > 0, \tag{4.8}$$

and for k=1.11

$$g_2(x, \beta_i(b))(-1)^i > 0, \tag{4.9}$$

with $i \in \{1, 2\},\ x \in [s_2, s_1]$.
Then

$$d_L(L + N, \Omega_5) = 1 \pmod 2,$$

where

$$\Omega_5 = \{x \in C^1(J) : s_2 < x(t) < s_1,\ \beta_1(t) < x'(t) < \beta_2(t) \text{ for all } t \in J, \\ \exists t_x \in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}.$$

Corollary 14. *Suppose that there exist real numbers* $r_1 > r_2$, $c_1 < 0 < c_2$, *such that* (4.2) *and* (4.3) *are satisfied for all* $(t, x) \in J \times [r_2 + c_1(b-a), r_1 + c_2(b-a)]$. *If* k=1.11, *we suppose that* (4.4), (4.5) *are satisfied for* $x \in [r_2 + c_1(b-a), r_1 + c_2(b-a)]$.
Then

$$d_L(L + N, \Omega_6) = 1 \pmod 2,$$

where

$$\Omega_6 = \{x \in C^1(J) : r_2 + c_1(b-a) < x(t) < r_1 + c_2(b-a), \\ c_1 < x'(t) < c_2, \forall\, t \in J.\}$$

Example 15. Suppose $f_1, f_2, f_3 \in C(J), k, m \in \mathbf{N}$. The function

$$f(t, x, y) = f_1(t)x^{2k+1} + f_2(t)y^{2m+1} + f_3(t)$$

satisfies the conditions of Corollary 12, if $f_1, f_2 > 0$ on J, and it satisfies the conditions of Corollary 14, if $f_1 < 0,\ f_2 > 0$ on J and either $m > k$ or $m = k$, $f_2(t) > \|f_1\|_{\max}(b-a)^{2k+1}$ for all $t \in J$.

Other type of conditions which can be used instead of (2.1) in Theorem 1 and Theorem 2 are one-sided growth conditions which were used by Kiguradze [5] in some existence theorems.

1. The one-sided Bernstein-Nagumo condition:

$$\begin{gathered}\exists\,\omega \in C(\mathbf{R}_+),\ \omega \text{ positive}, \ \int_0^\infty \frac{ds}{\omega(s)} = \infty \text{ and} \\ f(t,x,y) \le \omega(|y|)\cdot(1+|y|) \\ \forall (t,x) \in J \times [\sigma_1(t), \sigma_2(t)] \times \mathbf{R}.\end{gathered} \tag{4.10}$$

2. The one-sided linear growth condition:

$$\begin{gathered}\exists a_1, a_2 \in (0,\infty), \rho \in C(J\times\mathbf{R}), \text{ non-negative and non-decreasing} \\ \text{in the second argument such that} \\ f(t,x,y) \le a_1|x| + a_2|y| + \rho(t,|x|+|y|) \\ \forall (t,x,y) \in J \times \mathbf{R}^2,\end{gathered} \tag{4.11}$$

where

$$a_1(b-a)^2 + a_2(b-a) < 1$$

and

$$\lim_{z\to\infty} \frac{1}{z}\int_a^b \rho(t,z)dt = 0.$$

Note 16. Let us remember that if f satisfies (4.11) it satisfies (4.10) as well.

For the proof of the following theorems we need lemmas on a priori estimates for solutions of the problems (1.1), (k), k$\in \{1.5, 1.7, 1.11\}$.

Lemma 17. *Suppose*

$$\sigma_1(t) < \sigma_2(t) \quad \textit{for all } t \in J.$$

Let (4.10) *be satisfied. If* k=1.11, *suppose moreover*

$$\lim_{y\to\infty} g_1(r_2,y) > 0, \ \lim_{y\to-\infty} g_2(r_2,y) < 0, \tag{4.12}$$

$r_1 = min\{\sigma_1(t) : t \in J\}$, $r_2 = max\{\sigma_2(t) : t \in J\}$.
Then there exists $\mu^* \in (0,\infty)$ *such that for any solution* u *of the problem* (1.1), (k), *the implication*

$$\sigma_1(t) < u(t) < \sigma_2(t) \ \textit{on } J \Longrightarrow \|u'\|_{\max} < \mu^*$$

is valid.

Lemma 18. *Let $r_1, r_2 \in \mathbf{R}, r_1 < r_2$ and let* (4.11) *be satisfied. If* k=1.11, *suppose moreover*

$$\lim_{y\to\infty} g_1(x,y) > 0, \lim_{y\to-\infty} g_2(x,y) < 0, \tag{4.13}$$

uniformly for $x \in \mathbf{R}_+$.
Then there exists $\nu^* \in (0,\infty)$ such that for any solution u of the problem (1.1), (k), *the implication*

$$\exists t_u \in J : r_1 < u(t_u) < r_2 \Longrightarrow \|u'\|_{\max} < \nu^*$$

is valid.

Theorem 19. *Let* (4.10) *be fulfilled and let*

$$\sigma_1(t) < \sigma_2(t) \text{ for all } t \in J.$$

If k=1.11, *suppose moreover* (4.12).
Then there exists $r^* \in (0,\infty)$ such that

$$d_L(L+N, \Omega_6) = 1 \pmod 2,$$

where

$$\Omega_6 = \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t)\ \forall t \in J,\ \|x'\|_{\max} < r^*\}.$$

Theorem 20. *Let* (4.11) *be fulfilled and let*

$$\sigma_2(t) < \sigma_1(t) \text{ for all } t \in J.$$

If k=1.11, *suppose moreover* (4.13).
Then there exists $r^* \in (0,\infty)$ such that

$$d_L(L+N, \Omega_7) = 1 \pmod 2,$$

where

$$\Omega_7 = \{x \in C^1(J) : \|x\|_{\max} + \|x'\|_{\max} < r^*,\ \exists t_x \in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}.$$

5 Multiplicity results for f unbounded

We can extend the results of the Section 3 onto differential equations with an unbounded right-hand side $f \in C(J \times \mathbf{R}^2)$. We will present here such extension of some multiplicity results.

Let us suppose that σ_1, σ_2 and σ_3 are strict lower, upper and lower solutions of (1.1), (k), k∈{1.5,1.7,1.11}. Using Theorem 11 and Theorem 13 we get the following multiplicity result:

Theorem 21. *Suppose that* (3.1), (4.6) *and, according to* k, *the condition* (4.7) *or* (4.8) *or* (4.9) *are fulfilled for all* $(t,x) \in J \times [\sigma_1(t), s_3]$, *where* $s_3 = \max\{\sigma_3(t) : t \in J\} + \int_a^b \gamma(t)\,dt$.
Then the assertion of Theorem 8 *is valid.*

Similarly, by means of Theorem 19 and Theorem 20 and the fact that (4.11) and (4.13) are the special cases of (4.10) and (4.12), we get:

Theorem 22. *Let us suppose that* (3.1) *and* (4.11) *are fulfilled and, for* k=1.11, *suppose moreover* (4.13). *Then the assertion of Theorem* 8 *is valid.*

Now, let us consider the dual situation, where σ_3 is an upper solution of (1.1), (k).

Theorem 23. *Suppose that* (3.2), (4.6) *and, according to* k, *the condition* (4.7) *or* (4.8) *or* (4.9) *are fulfilled for all* $(t,x) \in J \times [b_3, \sigma_2(t)]$, *where* $b_3 = \min\{\sigma_3(t) : t \in J\} - \int_a^b \gamma(t)\,dt$.
Then the assertion of Theorem 9 *is valid.*

Theorem 24. *Let us suppose that* (3.2) *and* (4.11) *are fulfilled and, for* k=1.11, *suppose moreover* (4.13). *Then the assertion of Theorem* 9 *is valid.*

For constant lower and upper solutions we can generalize the theorems from [11], concerning the multiplicity results of the Ambrosetti-Prodi type for the periodic problem.

Theorem 25. *Suppose* k$\in\{$1.5,1.7$\}$. *Let* $n \in \mathbf{N}$, $n \geq 2$, $c_1, c_2, s_1, r_1, \ldots, r_{n+1} \in \mathbf{R}$, $c_1 < 0 < c_2$, *satisfy* (3.3), (3.4), (4.3) *for all* $(t,x) \in J \times [r_1, r^*]$, *where*

$$r^* = \begin{cases} r_{n+1} & \text{for } n \text{ odd} \\ r_{n+1} + \max\{|c_1|, c_2\}(b-a) & \text{for } n \text{ even.} \end{cases} \tag{5.1}$$

Then there exist $s_2, s_3 \in (-\infty, s_1)$, $s_3 \leq s_2$, *such that the problem* (3.5) *has:*

(i) *at least* n *different solutions* u_i, $i = 1, .., n$, *satisfying*

$$r_1 < u_i(t) < r^* \text{ for all } t \in J, i \in \{1, \ldots, n\}; \tag{5.2}$$

(ii) *at least* $\frac{n+1}{2}$ $(\frac{n}{2})$ *solutions satisfying* (5.2) *for* $s = s_2$ *and* n *odd (even);*

(iii) *provided* $s_3 < s_2$ *at least one solution satisfying* (5.2) *for* $s \in [s_3, s_2)$;

(iv) *no solution satisfying* (5.2) *for* $s < s_3$.

Theorem 26. *Suppose* k$\in\{$1.5,1.7$\}$. *Let* $n \in \mathbf{N}$, $n > 2$, *be odd and let further* $s_1, r_1, \ldots, r_{n+1} \in \mathbf{R}$ *satisfy* (3.3) *and* (3.4). *Further, let* (4.10) *be fulfilled.*
Then there exists $r^* \geq r_{n+1}$ *such that* (i)–(iv) *of Theorem* 25 *are valid.*

Theorem 27. *Suppose* k$\in\{$1.5,1.7$\}$. *Let* $n \in \mathbf{N}$, $n \geq 2$, *be even and let further* $s_1, r_1, \ldots, r_{n+1} \in \mathbf{R}$ *satisfy* (3.3) *and* (3.4). *Further let* (4.11) *be fulfilled.*
Then there exists $r^* \geq r_{n+1}$ *such that* (i)–(iv) *of Theorem* 25 *are valid.*

Note 28. Close results concerning the existence of two or three solutions of the periodic problem can be found also in [1] and [13].
For f satisfying the Carathéodory conditions on $J \times \mathbf{R}^2$ the results of Corollary 3, Theorem 4 and Theorem 6 can be proven as well. The multiplicity results of the Theorems 8–10 and the theorems for f unbounded of the Sections 4 and 5 have to be a little modified because in the Carathéodory case solutions can interact strict lower and upper solutions.

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