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Algebraic elements in the transformation theory of 2nd order linear oscillatory differential equations

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ALGEBRAIC ELEMENTS IN THE TRANSFORMATION THEORY OF 2nd ORDER LINEAR OSCILLATORY DIFFERENTIAL EQUATIONS

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1. In the last fifteen years, I have developed a transformation theory of ordinary 2nd order linear homogeneous differential equations in the real domain. It is a qualitative theory of global character. This theory deals with the effect of processes connected with the transformations of the variables on the integrals of the mentioned differential equations.

The origin of the transformation theory of 2nd order linear differential equations is due to E. E. KUMMER, who was the first to find the 3rd order non-linear differential equation which forms the basis of the transformation theory (1834). This equation is:

$$\text{(Qq)} \qquad -\{X, t\} + Q(X)\, X'^2 = q(t);$$

Q and q are given functions of a variable, X the unknown function and the symbol $\{X, t\}$ denotes the Schwarz derivative of X at the point t:

$$\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\frac{X''^2(t)}{X'^2(t)}.$$

Kummer's ideas have prepared the way for more extensive investigations into the transformations of linear differential equations of the nth order in connection with the equivalence problem. The most important results in this field are due to E. LAGUERRE, F. BRIOSCHI, G. H. HALPHEN, A. R. FORSYTH, S. LIE and P. APPELL, in whose works we occasionally also find information about transformations of 2nd order differential equations in the complex domain.

The transformation theory in the real domain which I have developed may perhaps at first sight appear only as a special case of the linear differential equations of the nth order ($n \geq 2$). One is, nevertheless, necessarily led to a systematic treatment of this case $n = 2$. This is due to the fact that the linear differential equations of the 2nd order not only occupy a special position among those of the n (≥ 2)$^{\text{th}}$ order, since only in case of $n = 2$ two differential equations are always equivalent, but the results concerning transformations of 2$^{\text{nd}}$ order differential equations are most useful even for a general n. A systematic investigation of this special case leads, moreover, to a considerable enrichment of the classical theory of the 2$^{\text{nd}}$ order differential equations, both as to the formation of new notions and as to the development of the method.

2. The kernel of the mentioned transformation theory of 2$^{\text{nd}}$ order differential equations consists in investigating the connections between the solutions of the 2$^{\text{nd}}$ order linear differential equations

$$\text{(q)} \qquad y'' = q(t)\, y, \qquad \ddot{Y} = Q(T)\, Y \qquad \text{(Q)}$$

and Kummer's non-linear 3$^{\text{rd}}$ order differential equations (Qq), (qQ). The functions q, Q, which I shall occasionally call *carriers* of the differential equations (q), (Q), are generally only supposed to be continuous in their (open) intervals of definition $j = (a, b)$, $J = (A, B)$. A fundamental piece of information about the mentioned connections, which was already known to Kummer, is that the solutions $X(t)$, $x(T)$ of the differential equations (Qq), (qQ) transform all the integrals Y, y of the linear differential equations (Q), (q), in the sens of the following formulas:

$$\text{(1)} \qquad y(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \qquad Y(T) = \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}}.$$

3. Let us now first introduce some basic notions essetial to any further research into the transformation theory in question.

Consider a differential equation (q) in an (open) interval $j = (a, b)$. The carrier q is only assumed to be continuous. The *integral space* r of the differential equation (q) is understood to be the set of all the integrals of (q). The *basis* (u, v) *of the differential equation* (q) stands for a sequence of two linearly independent integrals u, v of (q). The basis of the *integral space* r is a basis of the differential equation (q).

One of the most important notions of the transformation theory is the notion of a phase, about which I shall now say a few words.

We discern phases of a *basis* (u, v) of the differential equation (q) and phases of the *differential equation* (q).

By a *phase of the basis* (u, v) of the differential equation (q) we mean any function α continuous in the interval j and satisfying in the latter, except for the zeros of the integral v, the equation $\operatorname{tg} \alpha(t) = u(t) : v(t)$.

It is easily understood that the phases of the basis (u, v) form a countable system, the so called *phase-system* of the basis (u, v) and that the singular phases of the system differ by integer multiples of the number π.

A phase of the *differential equation* (q) is understood to be a phase of any basis of the differential equation (q).

Every phase α of the differential equation (q) has, in the interval j, the following properties:

$$1. \quad \alpha \in C^3, \qquad 2. \quad \alpha' \neq 0.$$

By means of a phase α of the differential equation (q), the carrier q of the latter is uniquely defined, in the sense of the formula

$$q(t) = -\{\alpha, t\} - \alpha'^2(t). \tag{2}$$

The notion of a phase is closely connected with that of a phase function:

A *phase function* in the interval j is understood to be a function with the above properties 1., 2. A phase function α is a phase of the differential equation (q) with the carrier q defined in the sense of the formula (2).

A phase function α is called *elementary* if its values at any two points t, $t + \pi \in j$ are connected in the following way: $\alpha(t + \pi) = \alpha(t) + \pi \,.\, \operatorname{sgn} \alpha'$.

The phases I have spoken about are the so called first phases of the basis (u, v) or the differential equation (q). Besides these, one analogously defines the second phases, namely by means of the equation $\operatorname{tg} \beta(t) = u'(t) : v'(t)$. Since we shall, in what follows, not deal with the latter, we shall simply always refer to phases instead of first phases.

4. Let us now restrict our consideration to oscillatory differential equations (q). The term "oscillatory" means that the integrals of the differential equation (q) vanish, infinitely many times, in both directions towards the endpoints a, b of the interval $j = (a, b)$.

We shall start our considerations with the theorem that the differential equation (q) is oscillatory if, and only if, its phases are unbounded on both sides, from above and from below.

The phases α of an oscillatory differential equation (q) have, therefore, besides the properties 1. and 2., even the following one:

$$3. \quad \lim_{t \to a+} \alpha(t) = -\infty \,.\, \operatorname{sgn} \alpha', \qquad \lim_{t \to b-} \alpha(t) = \infty \,.\, \operatorname{sgn} \alpha'.$$

We see that a phase function unbounded on both sides is a phase of an oscillatory differential equation (q), i.e. the one with the carrier q defined in the sens of formula (2).

Oscillatory differential equations (q) have, furthermore, the characteristic property that they allow, in their intervals of definition, certain privileged functions, i.e. the so called central dispersions $\ldots, \varphi_{-2}(t), \varphi_{-1}(t), \varphi_0(t), \varphi_1(t), \varphi_2(t), \ldots.$ The *central dispersion with the index* $\nu = 0, \pm 1, \pm 2, \ldots$ of the differential equation (q) is understood to be the function $\varphi_\nu(t)$ defined in the interval j as follows:

The value $\varphi_n(t)$ or $\varphi_{-n}(t)$ of the central dispersion φ_n or φ_{-n} ($n = 1, 2, \ldots$) is, at every point $t \in j$, the n^{th} number conjugated with t on the right or on the left with regard to the differential equation (q). In other words: If one considers an integral y of the differential equation (q), vanishing at the point t, then $\varphi_n(t)$ or $\varphi_{-n}(t)$ is the n^{th} zero of y on the right or on the left of t. $\varphi_0(t)$ stands for the function t. The function φ_1 is also called the *fundamental dispersion* of the differential equation (q) and is briefly denoted by φ.

Every central dispersion φ_ν has, in the interval j, the following properties:

1. $\varphi_\nu(t) > \varphi_{\nu-1}(t)$, 2, $\varphi_\nu \in C^3$, 3. $\varphi_\nu'(t) > 0$, 4. $\lim\limits_{t \to a+} \varphi_\nu(t) = -\infty$, $\lim\limits_{t \to b-} \varphi_\nu(t) = \infty$

We see that every central dispersion φ_ν is an increasing phase function, unbounded on both sides.

Moreover, we can show that:

Every central dispersion φ_ν and every phase α of the differential equation (q) are connected, at every point $t \in j$ by the so called *Abelian relation*

$$\alpha\varphi_\nu(t) = \alpha(t) + \nu\pi \,.\, \operatorname{sgn} \alpha'. \tag{3}$$

Instead of $\alpha[\varphi_\nu(t)]$ we simply write $\alpha\varphi_\nu(t)$.

Forming, in (3), on both sides the Schwarz derivative, one receives, with regard to (2), the relation

$$-\{\varphi_\nu, t\} + q(\varphi_\nu)\, \varphi'^2_\nu = q(t).$$

We see that every central dispersion φ_ν satisfies Kummer's differential equation (qq) and. consequently, transforms every integral Y of the differential equation (q) into an integral y of the same differential equation (q) in the sense of formula (1).

The central dispersion φ_ν are the so called central dispersions of the first kind of the differential equation (q). Besides these, one also definies central dispersions of the 2^{nd}, 3^{rd} and 4^{th} kind of the differential equation (q). In what follows we shall, however, not meet with the latter and will therefore simply refer only to central dispersions instead of to central dispersions of the 1^{st} kind.

5. Let us now make a closer study of the transformation theory of oscillatory differential equations and, for this purpose, first briefly describe a constructive integration theory of Kummer's differential equation (Qq): One first defines, constructively, certain functions continuously dependent on three parameters, i.e. the so called *general dispersions* of the differential equations (Q), (q), and then shows that the latter are exactly the integrals of Kummer's differential equation (Qq).

Let, then, (q), (Q) be arbitrary oscillatory differential equations in the intervals $j = (a, b)$, $J = (A, B)$. Their integral spaces will be denoted by r or R.

Let $t_0 \in j$, $T_0 \in J$ be arbitrary numbers. Choose in the integral space r a basis (u, v) and in the integral space R a basis (U, V) such that

$$u(t_0)\, V(T_0) - v(t_0)\, U(T_0) = 0. \tag{4}$$

It is easily understood that the choice of the latter depends on two arbitrary parameters. Let us now define, by means of the bases (u, v), (U, V), a linear representation p of the integral space r on the integral space R by making correspond, to every integral $y \in r$ of (q), $y = \lambda u + \mu v$, the integral $py = Y = \lambda U + \mu V$ of (Q), formed with the same constants λ, μ. The quotient $\chi p = w : W$ of the wronskians w or W of the basis (u, v) or (U, V) is allcalled the *characteristic* of the linear representation p. The latter has, with regard to the relation (4), the following property: Every integral $y \in r$ of (q), vanishing at t_0, is in the linear representation p represented on an integral $Y \in R$ of (Q), vanishing at T_0. In other words: $y(t_0) = 0$ always yields $py(T_0) = 0$ With regard to this property, we call the linear representation p *normalized with respect to the numbers* t_0, T_0.

Let us, moreover, consider the numbers conjugated, both on the left and on the right, with t_0, with respect to the differential equation (q): $\ldots, t_{-2} = \varphi_{-2}(t_0)$, $t_{-1} = \varphi_{-1}(t_0)$, $t_0 = \varphi_0(t_0)$, $t_1 = \varphi_1(t_0)$, $t_2 = \varphi_2(t_0)$, $\ldots$, and, similarly, the analogous numbers with respect to the differential equation (Q): $\ldots$, $T_{-2} = \Phi_{-2}(T_0)$, $T_{-1} = \Phi_{-1}(T_0)$, $T_0 = \Phi_0(T_0)$, $T_1 = \Phi_1(T_0)$, $T_2 = \Phi_2(T_0)$, $\ldots$. Every interval $j_\nu = [t_\nu, t_{\nu+1})$ or $j'_\nu = (t_{\nu-1}, t_\nu]$ for $\nu = 0, \pm 1, \pm 2, \ldots$ is called the ν^{th} *right* or *left-hand side basic interval* of the differential equation (q) with respect to the number t_0; the intervals $J_\nu = [T_\nu, T_{\nu+1})$ or $J'_\nu = (T_{\nu-1}, T_\nu]$ are called analogously. We see: every number $t \in j$ lies in a determined basic interval j_ν or j'_ν and, vice versa, every basic interval j_ν or j'_ν contains exactly one zero of every integral of (q). An analogous statement holds, of course, for every number $T \in J$ and for every integral Y of (Q).

Now we shall define, in the interval j, a function X as follows:

Let $t \in j$ be an arbitrary number and y an integral of the differential equation (q) vanishing at the point t. The number t lies in a determined right-hand side ν^{th} basic interval j_ν.

The value $X(t)$ of the function X at the point t is, according to whether $\chi p > 0$ or $\chi p < 0$, given as follows:

In case $\chi p > 0$, $X(t)$ is a zero of the integral py of (Q), namely the one lying in the right-hand side ν^{th} basic interval J_ν.

In case of $\chi p < 0$, $X(t)$ is a zero of the integral py of (Q), namely the one lying in the left-hand side $-\nu^{\text{th}}$ basic interval $J'_{-\nu}$.

The function X is called *general dispersion of the differential equations* (q), (Q) *with respect to the numbers* t_0, T_0 *and the linear representation* p. At the point t_0 it obviously takes on the value $T_0 : X(t_0) = T_0$.

It is obvious that the general dispersions we have just defined continuously depend on three arbitrary parameters: one is the arbitrarily chosen initial value T_0 and the two others are the parameters of the corresponding normalized linear representation p.

From the properties of the general dispersions, which can be deduced from the above construction, we shall only mention the following:

Let X be a general dispersion of the differential equations (q), (Q) and p the corresponding linear representation of the integral space r on the integral space R.

1. The set of values of the function X is the interval $J : X(j) = J$.

2. The function X is a phase function.

3. There holds $\operatorname{sgn} X' = \operatorname{sgn} \chi p$. Consequently, the function X increases or decreases according to whether $\chi p > 0$ or $\chi p < 0$.

4. The function X may be expressed by means of two phases $\alpha(t)$, $A(T)$ of the differential equations (q) or (Q) in the following way:

$$X(t) = A^{-1}\alpha(t).$$

Vice versa, the function $A^{-1}\alpha(t)$ formed by means of arbitrary phases α, A of the differential equations (q) or (Q) is a general dispersion of the differential equations (q), (Q).

Moreover, there holds the following theorem:

5. The general dispersions of the differential equations (q), (Q) are exactly the integrals of Kummer's differential equation (Qq).

6. The above considerations, and especially the constructive integration theory we have just outlined, hold for differential equations (q), (Q) in arbitrary (open) intervals j, J. Let us now restrict our considerations to the case $j = J = (-\infty, \infty)$ and, consequently, deal only with oscillatory differential equations (q), (Q) in the interval $j = (-\infty, \infty)$. That is exactly the case when algebraic elements enter the transformation theory and algebraic theorems, particularly those from the group theory, allow us to learn new facts about the integrals of Kummer's differential equation (Qq).

The prototype of the differential equations to be considered is the differential equation (-1), i.e. $y'' = -y$ in the interval $j = (-\infty, \infty)$. The integrals of this differential equation obviously have, in both directions, an infinite number of π-equidistantly displaced zeros, i.e. arranged so that the difference between any two neighbouring zeros of every integral is always the same, namely π. Hence it follows that the fundamental dispersion φ of the differential equation

(-1) is linear, $\varphi(t) = t + \pi$, and more generally, that the following formula holds for the central dispersion φ_ν:

$$\varphi_\nu(t) = t + \nu\pi \qquad (\nu = 0, \pm 1, \pm 2, \ldots).$$

If, furthermore, α is a phase of the differential equation (-1), then the Abelian relation (3) yields

$$\alpha(t + \pi) = \alpha(t) + \pi \,.\, \operatorname{sgn} \alpha'.$$

We see that all the phases of the differential equation (-1) are elementary.

7. Let us now consider the set $\mathfrak{G}$ formed of all the phase functions-unbounded on both sides, i.e. both from above and from below- in the interval $j = (-\infty, \infty)$. We see, first, that the function $\alpha\beta(t)$ composed of two arbitrary elements α, $\beta \in \mathfrak{G}$, is again an element of $\mathfrak{G}$. With regard to this, we shal now introduce, in the set $\mathfrak{G}$, a multiplication consisting in composing functions. For any two phase functions α, $\beta \in \mathfrak{G}$, the product $\alpha\beta$ is therefore understood to be the composed function $\alpha[\beta(t)]$. The set $\mathfrak{G}$ is obviously, with regard to this multiplication, a semi-group. The latter evidently contains the unit element 1, i.e. the phase function $\varepsilon(t) = t$; furthermore, there exists, to every element $\alpha(t) \in \mathfrak{G}$, the inverse element $\alpha^{-1}(t)$, namely the function $\alpha^{-1}(t)$ inverse to the function $\alpha(t)$. Thus we have shown that the set $\mathfrak{G}$, together with the considered multiplication, forms a group. Let us call it the *phase group* $\mathfrak{G}$.

The phase group $\mathfrak{G}$ consists, according to its definition, exactly of the phases of all the oscillatory differential equations (q) in the interval $j = (-\infty, \infty)$. To discern the phases of the singular differential equations (q), we shall now introduce, in the phase group $\mathfrak{G}$, a relation $\mathscr{R}$ in the following way: the relation $\alpha \mathscr{R} \beta$ expresses that the phase function β is a phase of the same differential equation (q) as the phase function α. It is easily verified that this relation $\mathscr{R}$ is reflexive, symmetrical and transitive and therefore forms an equivalence relation. Consequently, there exists, on the phase group $\mathfrak{G}$, a decomposition $\bar{R}$ such that any two elements α, $\beta \in \mathfrak{G}$ are phases of the same differential equation (q) if, and only if, they lie in the same element $\bar{a} \in \bar{R}$.

Let now $\mathfrak{E}$ be that element of $\bar{R}$ in which the unit element $\varepsilon(t) = t$ of $\mathfrak{G}$ is contained. The formula (2) shows that the phase function $\varepsilon(t)$ is a phase of the above differential equation (-1). Consequently, the element $\mathfrak{E} \in \bar{R}$ consists of all the phases of the differential equation (-1) and can be shown to be an undergroup of $\mathfrak{G} : \mathfrak{E} \subset \mathfrak{G}$. This undergroup will be called the *fundamental undergroup* of $\mathfrak{G}$. Furthermore, there holds the following theorem: *The decomposition $\bar{R}$ coincides with the right-hand side class decomposition of the phase group $\mathfrak{G}$ with regard to $\mathfrak{E}$:*

$$\bar{R} = \mathfrak{G}/_r\mathfrak{E}.$$

The set of all the oscillatory differential equations (q) in the interval $j = (-\infty, \infty)$ therefore admits an one-one representation on the right-hand side class decomposition $\mathfrak{G}/_r\mathfrak{E}$, namely the one that makes correspond, to every differential equation (q), the element $\bar{q} \in \mathfrak{G}/_r\mathfrak{E}$ consisting of the phases of (q).

We shall now consider the undergroup of $\mathfrak{G}$ consisting of all the *elementary*, phase functions; let us denote it by $\mathfrak{H}$. Since, as we know, all the phases of the differential equation (-1) are elementary and form the fundamental undergroup $\mathfrak{E}$, we see, first, that $\mathfrak{H}$ is an overset of $\mathfrak{E}$. A further investigation which I cannot describe here in detail, shows that the elementary phase functions generally depend on arbitrary periodic functions with period π whereas the elements of $\mathfrak{E}$ depend only on three parameters. It follows that $\mathfrak{H}$ is a *proper* overset of $\mathfrak{E}$. It can, moreover, be shown that $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$. Hence there hold, between the groups $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{E}$, the relations:

$$\mathfrak{G} \supset \mathfrak{H} \supset \mathfrak{E}, \tag{5}$$

the overgroups as well as the subgroups in question being proper.

Let us now consider the righ-hand side class decomposition $\bar{H}$ of the phase group $\mathfrak{G}$ with respect to the subgroup $\mathfrak{H} : \bar{H} = \mathfrak{G}/_r\mathfrak{H}$.

First, the relations (5) yield the formula:

$$(\bar{H} =)\ \mathfrak{G}/_r\mathfrak{H} \geq \mathfrak{G}/_r\mathfrak{E}\ (= \bar{R}),$$

by which the decomposition $\bar{H}$ is a covering of $\bar{R}$, in other words, each element of $\bar{H}$ is the set-sum of some elements of $\bar{R}$ ([1]). Furthermore, the following theorem applies:

The elements of $\bar{R}$, contained in an arbitrary element $\mathfrak{H}\alpha \in \bar{H}$ $(\alpha \in \mathfrak{G})$, consist of phases of all the differential equations (q) whose fundamental dispersion φ is the same.

Finally, let us note that cardinal number of the set of the elements of $\bar{R}$ contained in an arbitrary element $\mathfrak{H}\alpha \in \bar{H}$ is always the same and equal to that of the continuum. Consequently: the cardinal number of the set of all the differential equations (q) whose fundamental dispersion φ is the same does not depend on the latter and is always equal to the cardinal number $\aleph$ of the continuum ([2]).

8. We shall now return to the general dispersions of two differential equations (q), (Q), namely to the integrals of Kummer's differential equation (Qq). As we have said above, every general dispersion X of the differential equations (q), (Q) transforms all the integrals Y of the differential equation (Q) to integrals y of the differential equations (q), the transformation being expressed by the first formula (1).

It can, first, be easily seen that the general dispersions of the differential equations (q), (Q) form elements of the phase group $\mathfrak{G}$. Indeed, every general dispersion X of (q), (Q) is, as we know, a phase function, whose set of values coincides with the interval of definition J of (Q). But since $J = (-\infty, \infty)$, the general dispersion X is a phase function unbounded on both sides and hence an element of the phase group $\mathfrak{G}$. It is, besides, easy to show that the general dispersion X is a phase of the differential equation (q_X), the relation between the functions q_X, q, Q being as follows:

$$q_X(t) = q(t) - [1 + Q(X)]\, X'^2(t).$$

We shall now determine the general dispersions of the differential equations (q), (Q) in the phase group $\mathfrak{G}$ by means of the following theorem:

Let α be a phase of the differential equation (q) *and A be one of* (Q). *The integral space $(X)_{(Qq)}$ of Kummer's differential equation* (Qq), *i.e. the set of all general dispersions of the differential equations* (q), (Q) *is given by the following formula*:

(6) $$(X)_{(Qq)} = A^{-1}\mathfrak{E}\alpha;$$

$\mathfrak{E}$ *naturally stands for the fundamental subgroup of* $\mathfrak{G}$.

This theorem yields a number of results of which I shall only mention a few, so as not to spoil the general outline by too many details.

It may first be shown [by means of (6)], that the integral spaces $X_{(Qq)}$, $(X)_{(Q_1q_1)}$ of two arbitrary Kummer's differential equations (Qq), (Q_1q_1) have the same cardinal numbers and can be one-one represented on each other in the sense of formula:

$$X_1 = Z^{-1}Xz.$$

In this formula: $X \in (X)_{(Qq)}$, $X_1 \in (X)_{(Q_1q_1)}$, Z standing for a fixed integral of (QQ_1) and z for one of (qq_1).

Let us next consider the case of two *coinciding* differential equations (Q), (q) and Kummer's corresponding differential equation (qq). Every integral of this differential equation transforms, in the sense of formula (1), every integral Y of the differential equation (q) into an integral y of the same differential equation (q). From the above theorem it follows that:

The integral space $(X)_{(qq)}$ of Kummer's differential equation (qq) is the subgroup of $\mathfrak{G}$ conjugated with $\mathfrak{E}$:

(7) $$(X)_{(qq)} = \alpha^{-1}\mathfrak{E}\alpha;$$

α naturally denotes an arbitrary phase of the differential equation (q).

Consequently:

The integral spaces $(X)_{(qq)}$, $(X)_{(q_1q_1)}$ of two arbitrary differential equations (qq), (q_1q_1) are isomorphous, the isomorphism being given by the following formula:

(8) $$X_1 = z^{-1}Xz;$$

z denotes a fixed integral of the differential equation (qq_1).

A further consideration now permits to investigate, more closely, the algebraic structure of the integral space $(X)_{(qq)}$ of every differential equation (qq). One proceeds by first finding out the structure of the integral space of the differential equation $(-1, -1)$, i.e. of the group $\alpha^{-1}\mathfrak{C}\alpha$ $(\alpha \in \mathfrak{C})$ and then, by means by formula (8), passing to the differential equation (qq). One finds, particularly, that the increasing integrals contained in the integral space $(X)_{(qq)}$ of (qq) form a normal subgroup $\mathfrak{A}$ of index 2, the center of this subgroup coinciding with the infinite cyclic group formed by all the central dispersions φ_ν of the differential equation (q).

Herewith I have arrived at the conclusion of my lecture. Let me only add the remark, addressed particularly to those who take a special interest in the above considerations, that the latter form part of my book "Lineare Differentialtransformationen 2. Ordnung". This book will be published by the Deutscher Verlag der Wissenschaften, Berlin (DDR), in 1967.

BIBLIOGRAPHY

[1] Borůvka O.: *Grundlagen der Gruppoid- und Gruppentheorie. VEB Deutscher Verlag der Wiss., Berlin (DDR), 1960.*

[2] Borůvka O.: *Sur l'ensemble des équations différentielles linéaires ordinaires du deuxième ordre qui ont la même dispersion fondamentale. Bul. Instit. Polit. din Iaşi, IX (XIII), 1963, 11–20.*