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In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 301--305.

Persistent URL: http://dml.cz/dmlcz/700214

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

DISCRETISATION AND ERROR ESTIMATES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS OF THE FOURTH ORDER

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1. One of the problems arising in the application of the finite difference method in solving elliptic boundary value problems is the estimation of the discretization error. There exists an extensive literature for second order elliptic differential equations while there are only few papers dealing with higher order equations. The reason is that we have a very useful and simple tool for second order equations, the maximum principle, which holds both for differential equations and their finite difference analogs. There does not exist such a simple tool for higher order equations. In [1] I dealt with an elliptic equation of the fourth order a special case of which are the biharmonic equation and the equation for the deflection of orthotropic plates. In the paper there is described an $O(h^2)$ finite difference analog of the Dirichlet problem for this equation and an error estimate is proved but only for domains consisting of a finite number of rectangles the boundaries of which are a part of the mesh lines. In this lecture I will describe an $O(h^2)$ finite difference analog for domains of a general shape and will give estimates of the discretization error.

2. The equation considered is

$$(1) \qquad Lu \equiv \frac{\partial^2}{\partial x^2}\left(a(x, y)\frac{\partial^2 u}{\partial x^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\left(b(x, y)\frac{\partial^2 u}{\partial x\,\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(c(x, y)\frac{\partial^2 u}{\partial y^2}\right) = F(x, y)$$

(in fact, the method applies and the results remain true if we add to Lu an operator of the second order $Mu = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + qu$ where $x_1 = x$, $x_2 = y$ and $\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq 0$, $q \geq 0$). We assume that

(2) $\quad a(x, y) > 0, \quad c(x, y) > 0, \quad 0 \leq b(x, y) \leq 2 \min [a(x, y), c(x, y)].$

Let $\mathscr{D}$ be the domain in which the equation is considered and let $\dot{\mathscr{D}}$ be its boundary. The boundary conditions have the form

(3) $$D^p u = D^p f \qquad (p = 0, 1) \text{ on } \dot{\mathscr{D}},$$

where $D^0 u = u$, $D^1 u$ means any of the first derivatives and f is a given function defined in a domain $\Omega \supset \overline{\mathscr{D}}$. The coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$, the functions $f(x, y)$, $F(x, y)$ and the boundary $\dot{\mathscr{D}}$ are supposed so smooth that the solution $u(x, y)$ of the Dirichlet problem (1), (3) has bounded derivatives up to the sixth order inclusive.

To formulate the finite difference analog of the Dirichlet problem (1), (3) we cover the (x, y) plane in the usual manner by a square net formed by lines parallel to the axes. Let h be the corresponding mesh size. The mesh points will be denoted by (x, y) as any point in the (x, y) plane. The mesh functions, i.e. functions defined at mesh points will be denoted by $U(x, x)$, $E(x, y)$ etc. We use the usual notations

$$U_x(x, y) = h^{-1}[U(x + h, y) - U(x, y)], \quad U_{\bar{x}}(x, y) = h^{-1}[U(x, y) - U(x - h, y)],$$
$$U_{x\bar{x}} = h^{-2}[U(x + h, y) - 2U(x, y) + U(x - h, y)], \ \ldots$$

The operator Lu will be replaced by the difference operator

(4) $$L_h U = (aU_{x\bar{x}})_{x\bar{x}} + (bU_{xy})_{\bar{x}\bar{y}} + (bU_{\bar{x}\bar{y}})_{xy} + (cU_{y\bar{y}})_{y\bar{y}}$$

which represents an $O(h^2)$ approximation of Lu, i.e.

$$Lu - L_h u = O(h^2) \qquad \text{for } u \in C^6.$$

Let us introduce the sets $\mathscr{D}_h$, $\dot{\mathscr{D}}_h$ and $\dot{\mathscr{D}}_h^*$. By neighbors of a mesh point (x, y) we call 12 mesh points $(x + ih, y + jh)$ with $i, j = 0, \pm 1, \pm 2$, $1 \leq \leq i^2 + j^2 \leq 4$. Now $\mathscr{D}_h$ is the set of all mesh points from $\mathscr{D}$. $\dot{\mathscr{D}}_h$ is the set of neighbors of the mesh points from $\mathscr{D}_h$ which do not belong to $\mathscr{D}_h$, i.e. which do not lie in $\mathscr{D}$, $\dot{\mathscr{D}}_h^*$ is the set of mesh points from $\mathscr{D}_h$ such that at least one of their neighbors lies in $\dot{\mathscr{D}}_h$.

The discrete analog will be a mesh function defined on $\mathscr{D}_h$. First we set

(5) $$L_h U(x, y) = F(x, y), \qquad (x, y) \in \mathscr{D}_h - \dot{\mathscr{D}}_h^*.$$

To get the equations for the points $(x, y) \in \dot{\mathscr{D}}_h^*$ we will extrapolate the values $U(x, y)$, $(x, y) \in \dot{\mathscr{D}}_h$, by means of the boundary condition (3) and the values $U(x, y)$, $(x, y) \in \dot{\mathscr{D}}_h^*$ and we will insert these extrapolated values in the expression $L_h U$ formed formally. Consider first the point $(x - 2h, y)$. If it lies in $\dot{\mathscr{D}}_h$ the boundary $\dot{\mathscr{D}}$ intersects the segment $(x - 2h, y)$, (x, y) in a point $(x - \alpha h, y)$ with $0 < \alpha \leq 2$ and we set

$$U(x - 2h, y) = \left(\frac{2-\alpha}{\alpha}\right)^2 U(x, y) + 4\,\frac{\alpha - 1}{\alpha^2} f(x - \alpha h, y) -$$
$$- 2\,\frac{2-\alpha}{\alpha}\,h\,\frac{\partial F(x - \alpha h, y)}{\partial x}\,.$$

This is nothing else than an extrapolation of the second degree by means of the parabola assuming the value $U(x, y)$ in (x, y) and $u(x - \alpha h, y)$ in $(x - \alpha h, y)$ and having the derivate equal to $\frac{\partial u(x - \alpha h, y)}{\partial x}$ in $(x - \alpha h, y)$. If the point $(x - h, y)$ also belongs to $\dot{\mathscr{D}}_h$ then $0 < \alpha \leq 1$ and we set

$$U(x - h, y) = \left(\frac{1-\alpha}{1+\alpha}\right)^2 U(x + h, y) + \frac{4\alpha}{(1+\alpha)^2} f(x - \alpha h, y) -$$
$$- 2\,\frac{1-\alpha^2}{(1+\alpha)^2}\,h\,\frac{\partial F(x - \alpha h, y)}{\partial x}\,.$$

This time we use to the extrapolation the value of U in the point $(x + h, y)$ and again the given values of u and $\frac{\partial u}{\partial x}$ in $(x - \alpha h, y)$. It can happen that the point $(x + h, y)$ does not belong to $\mathscr{D}_h$. In this case the boundary $\dot{\mathscr{D}}$ intersects the segment $(x - 2h, y)$, $(x + h, y)$ at least twice and we extrapolate $U(x - 2h, y)$ and $U(x - h, y)$ by means of the values of u and $\frac{\partial u}{\partial x}$ in these intersections. Futher if the point $(x - h, y + h)$ belongs to $\dot{\mathscr{D}}_h$ we extrapolate the value $U(x - h, y + h)$ in the same way as in the first case, namely by means of the values u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ in the intersection $(x - \beta h, y + \beta h)$, $0 < \beta \leq 1$, of the boundary $\dot{\mathscr{D}}$ with the segment $(x - h, y + h)$, (x, y) and by means of $U(x, y)$. We have

$$U(x - h, y + h) = \left(\frac{1-\beta}{\beta}\right)^2 U(x, y) + \frac{2\beta - 1}{\beta^2} F(x - \beta h, y + \beta h) +$$
$$+ \frac{1-\beta}{\beta^2}\,h\left[-\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right]_{(x - \beta h, y + \beta h)}.$$

In this way we extrapolate all remaining values of $U(x, y)$ for $(x, y) \in \dot{\mathscr{D}}_h$. After inserting these values in the expression $L_h U(x, y)$ formed formally we get an expression of the form $\bar{L}_h U - l_h(f)$ where the operator $\bar{L}_h U$ contains the terms with $U(x, y)$, $(x, y) \in \mathscr{D}_h$, only and $l_h(f)$ consists of terms containing f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ ($l_h(f) = 0$ if $f \equiv 0$). We set

$$L_h(U) = F(x, y) + l_h(f), \qquad (x, y) \in \mathscr{D}_h^*. \tag{6}$$

3. Following [2] let us introduce the L_2 norms $||E||_0$, $||E||_1$, $||E||_2$. We set

$$||E||_0^2 = h^2 \sum_{S_h} E^2,$$
$$||E||_1^2 = ||E||_0^2 + ||E_x||_0^2 + ||E_y||_0^2,$$
$$||E||_2^2 = ||E||_0^2 + ||E_x||_1^2 + ||E_y||_1^2.$$

S_h means the set of all mesh points in the plane (x, y) and E is any mesh function defined on $\mathscr{D}_h$ and extended on S_h by setting $E(x, y) = 0$ for $(x, y) \notin \mathscr{D}_h$.

The main result is given by the following estimate: If E is the discretization error, i.e. $E(x, y) = u(x, y) - U(x, y)$ for $(x, y) \in \mathscr{D}_h$, $E(x, y) = 0$ for $(x, y) \notin \mathscr{D}_h$ then

$$||E||_2 = O(h^{\frac{3}{2}}). \tag{7}$$

There are good reasons to believe that this estimate cannot be improved, i.e. the exponent $\frac{3}{2}$ is the best though we use an $O(h^2)$ approximation. By means of the discrete SOBOLEV inequality it follows from (7)

$$\max_{(x,y) \in D_h} |E(x, y)| = O(h^{\frac{3}{2}}).$$

Futher by means of an inequality due to BRAMBLE (see [3], lemma 3.2) we get

$$\max_{(x,y) \in \mathscr{D}_h} (|E_x(x, y)| + |E_y(x, y)| = O\left(h^{\frac{3}{2}} . \lg \frac{1}{h}\right)$$

By means of another inequality due to BRAMBLE (see [3], lemma 3.3) it is easy to show that for $Lu = \Delta^2 u$ it follows from (7)

$$||E||_1 = O(h^2), \qquad \max_{(x,y) \in \mathscr{D}_h} |E(x, y)| = O\left(h^2 . \lg \frac{1}{h}\right).$$

4. For domains consisting of a finite sum of rectangles the boundaries of which are a part of the mesh lines it is possible to formulate the discrete analog in such a way that the discretization error satisfies

$$||E||_2 = O(h^2). \tag{8}$$

The asumptions are the same as in the general case with the exception of (2). It is sufficient to assume the uniform ellipticity. For simplicity let us consider a rectangle. The set of the mesh points lying inside the rectangle will be denoted by $\mathscr{D}_h$. Γ_h is the set of the mesh points lying on the boundary of the rectangle, $\overline{\Gamma}_h$ is the set of the mesh points lying outside of the rectangle at a distance h from the boundary. The mesh function U will be defined on the set $\mathscr{D}_h \cup \overline{\Gamma}_h \cup \Gamma_h$. In $\mathscr{D}_h$ we set

$$L_h U(x, y) = F(x, y), \qquad (x, y) \in \mathscr{D}_h.$$

On Γ_h we set $U = f$ and on $\overline{\Gamma}_h$ we extrapolate the value of U by means of the boundary values and two neighbors lying inside the rectangle. If, for instance, $(\xi, \eta) \in \Gamma_h$ and $(\xi - h, \eta) \in \overline{\Gamma}_h$ we set

$$U(\xi - h, \eta) = 3U(\xi + h, \eta) - \frac{1}{2}\, U(\xi + 2h, \eta) - \frac{3}{2} f(\xi, \eta) - 3h\, \frac{\partial f(\xi, \eta)}{\partial x}\,.$$

The estimate of the discretization error is given by (8) from which it follows

$$\max_{x,y \in \mathcal{G}_h} \quad |E(x, y)| = O(h^2)$$

$$\max \quad (|E_x(x, y)| + |E_y(x, y)| = O\left(h^2 \,.\, \lg \frac{1}{h}\right)\,.$$

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