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EXTENSION OF THE AVERAGING METHOD TO STOCHASTIC EQUATIONS

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This lecture was devoted to Ito's stochastic equations. These equations are usually written in the integral form

$$(1) \qquad x(t, \omega) = x_0(\omega) + \int_{t_0}^{t} a(\tau, x(\tau, \omega))\, d\tau + \int_{t_0}^{t} B(\tau, x(\tau, \omega))\, dw(\tau, \omega)$$

or in the equivalent differential form

$$(1') \qquad dx(t, \omega) = a(t, x(t, \omega))\, dt + B(t, x(t, \omega))\, dw(t, \omega)$$

The expressions $w(t, \omega)$ and $x(t, \omega)$ are random processes, i.e. there is given a triplet $(\Omega, \mathscr{F}, P)$ where Ω is a space, $\mathscr{F}$ is a σ-field of subsets of Ω and P is a probability measure which is defined on $\mathscr{F}$. All random processes or random values are $\mathscr{F}$-measurable functions of the parameter ω. Let R_n denote the n-dimensional Euclidean space. First the conditions are given under which the existence theorem holds:

1) Let $w(t, \omega)$ be a vector random process with stochastically independent increments and $F(t)$ a continuous function such that

$$E\,||w(t_2, \omega) - w(t_1, \omega)||^2 = F(t_2) - F(t_1), \qquad E(w(t_2, \omega) - w(t_1, \omega)) = 0$$

where E means the expectation.

2) There are given a vector function $a(t, x)$ and a matrix function $B(t, x)$ where x is also an n-dimensional vector. $a(t, x)$, $B(t, x)$ are continuous in both arguments and Lipschitz continuous in x:

$$||a(t, x) - a(t, y)|| \leqq K\,||x - y||, \qquad ||B(t, x) - B(t, y)|| \leqq K\,||x - y||.$$

3) There is given a random value $x_0(\omega)$ which is stochastically independent of all increments of $w(t, \omega)$ and $E||x_0(\omega)||^2 < \infty$.

Under these assumptions we can find the solution of (1) in the space $R_n \times \Omega$ of random processes $z(t, \omega)$ with the norm $\sqrt{E \sup_{\tau \in \langle 0, t\rangle} ||z(\tau, \omega)||^2}$. It is possible to prove this statement by means of the method of successive approximations, which converge in this space.

Now we can already pass to the average theory. Let us assume that the process $w_\varepsilon(t, \omega)$ and the function $F_\varepsilon(t)$ depend on a „small'' parameter ε for $\varepsilon \in \langle 0, \delta \rangle$ and that the following assumptions are fulfilled:

4) $w_\varepsilon^*(t, \omega) = w_\varepsilon(t, \omega) - w_0(t, \omega)$ is a process with stochasically independent increments again and $\lim_{\varepsilon \to 0} E \|w_\varepsilon^*(t_2, \omega) - w_\varepsilon^*(t_1, \omega)\|^2 = 0$ uniformly on every compact set of t_1, t_2

5) $\mathscr{F}_\varepsilon(t) \subset \mathscr{F}_0(t), \quad \mathscr{F}_\varepsilon(t) \subset \mathscr{F}_\varepsilon^*(t)$ or

5') $\mathscr{F}_0(t) \subset \mathscr{F}_\varepsilon(t), \quad \mathscr{F}_0(t) \subset \mathscr{F}_\varepsilon^*(t)$

where $\mathscr{F}_0(t), \mathscr{F}_\varepsilon(t), \mathscr{F}_\varepsilon^*(t)$ are the smallest σ-fields corresponding to $w_0(t, \omega)$, $w_\varepsilon(t, \omega)$, $w_\varepsilon^*(t, \omega)$.

6) $a(t, x, \varepsilon)$ depends on ε for $\varepsilon \in \langle 0, \delta \rangle$ such that K in 2) is independent of ε, there exists a continuous function $\psi(t)$ such that $\int_{t_1}^{t_2} \|a(t, 0, \varepsilon)\| \, dt \leqq \psi(t_2) - \psi(t_1)$ and a function $\varphi(\varepsilon) > 0$, $\varphi(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that

$$\left\| \int_{t_1}^{t_2} (a(\tau, x, \varepsilon) - a(\tau, x, 0)) \, d\tau \right\| \leqq \varphi(\varepsilon)\,(1 + \|x\|) \qquad \text{for } t_1 \leqq t_2 \leqq t_1 + 1.$$

7) $B(t, x, \varepsilon)$ depends on ε for $\varepsilon \in \langle 0, \delta \rangle$. The constant K in 2) is independent of ε,

$$\int_{t_1}^{t_2} \|B(t, 0, \varepsilon)\|^2 \, dF_\varepsilon(t) \leqq \psi(t_2) - \psi(t_1)$$

and

$$\int_{t_1}^{t_2} \|B(\tau, x, \varepsilon) - B(\tau, x, 0)\|^2 \, dF_\varepsilon(\tau) \leqq \varphi(\varepsilon)\,(1 + \|x\|^2)$$

for $t_1 \leqq t_2 \leqq t_1 + 1$, the functions $\varphi(\varepsilon)$, $\psi(t)$ being the same as in 6).

8) $x_0^{(\varepsilon)}(\omega)$ depends on ε for $\varepsilon \in \langle 0, \delta \rangle$ such that $x_0^{(\varepsilon)}(\omega)$ is stochastically independent of all increments of the processes $w_\varepsilon(t, \omega)$ and $w_0(t, \omega)$. The initial value $x_0^{(0)}(\omega)$ is stochastically independent of all increments of all the processes $w_\varepsilon(t, \omega)$ and $E\|x_0^{(\varepsilon)}(\omega) - x_0^{(0)}(\omega)\|^2 \to 0$ for $\varepsilon \to 0$.

Now ewerything is prepared to formulate the

Theorem 1. *Let the stochastic equations*

$$(2) \qquad x_\varepsilon(t, \omega) = x_0^{(\varepsilon)}(\omega) + \int_0^t a(\tau, x_\varepsilon(\tau, \omega), \varepsilon) \, d\tau + \int_0^t B(\tau, x_\varepsilon(\tau, \omega), \varepsilon) \, dw_\varepsilon(\tau, \omega)$$

be given and assumptions 1) *to* 8) *be fulfilled, then to every* $L > 0$ *and* $\eta > 0$ *there is* $\varepsilon_0 > 0$ *such that*

$$E \sup_{\tau \in \langle 0, L \rangle} \|x_\varepsilon(\tau, \omega) - x_0(\tau, \omega)\|^2 \leqq \eta \qquad \textit{for } 0 \leqq \varepsilon \leqq \varepsilon_0.$$

This result is very similar to a result of Gichman I. I. [1] which was unknown to me for a long time, since his work was not available. But Gichman's result was derived under different assumptions about the processes $w(t, \omega)$ pu the statement itself is also slightly different.

If we put $B(t, x) \equiv 0$ or $w(t, \omega) = \text{const.}$ in (1) then we obtain an ordinary differential equation and Theorem 1 is then the well-known theorem where the right-hand side of (2) fulfils condition 6). The stochastic part of (2) that is $B(t, x, \varepsilon)$ must fulfil condition 7) and that is stronger than a condition analogous to 6). The following example shows that the condition on B analogous to 6) would not be sufficient. Let x be a scalar and $w(t, \omega)$ the scalar Wiener proces i.e. the almost everywhere continuous process with stochastically independent increments for which $F(t) = t$. We shall consider the equation $x_\varepsilon(t, \omega) = \int_0^t \sin \frac{\tau}{\varepsilon} \, dw(\tau, \omega)$. By the well known theorem it holds $E|x_\varepsilon(t, \omega)|^2 = \int_0^t \sin^2 \frac{\tau}{\varepsilon} \, d\tau = \frac{t}{2} - \frac{\varepsilon}{4} \sin \frac{2t}{\varepsilon}$ and $\lim_{\varepsilon \to 0} E|x_\varepsilon(t, \omega)|^2 = \frac{t}{2}$ while $\int_0^t \sin \frac{\tau}{\varepsilon} \, d\tau \to 0$ for $\varepsilon \to 0$.

The number ε_0 in Theorem 1 depends on L. However, if we add some stability properties of solution of unperturbed equation i.e. of equation (2) for $\varepsilon = 0$ and if we omit the least upper bound in statement of Theorem 1 we can choose ε_0 independent of L. We shall use the concept of stability in average.

Definition 1. *The solution $\bar{x}(t, \omega)$ of (1′) is stable in average, if there is a function $\sigma(\eta) > 0$ such that $E||\bar{x}(t_0, \omega) - x(t_0, \omega)||^2 < \sigma(\eta)$ implies $E||\bar{x}(t, \omega) - x(t, \omega)||^2 < \eta$* for all $t \geqq t_0$.

Definition 2. *The solution $\bar{x}(t, \omega)$ of (1′) is asymptotically stable in average if it is stable in average and if there exist a number $A > 0$ and a function $T(\sigma, \eta)$ defined for $\sigma < A$, $\eta < A$ such that $E||\bar{x}(t_0, \omega) - x(t_0, \omega)||^2 < \sigma$ implies $E||\bar{x}(t, \omega) - x(t, \omega)||^2 < \eta$ for all $t \geqq t_0 + T(\sigma, \eta)$.*

Definition 3. *We say that the process $w(t, \omega)$ is homogeneous if all distributions $F_{t_1+h, t_2+h}(A) = P(w(t_2 + h) - w(t_1 + h) \in A)$ are independent of h (A is an arbitrary n-dimensional Borel set).*

Theorem 2. *Let the conditions* 1) *to* 8) *be fulfilled, let the convergence $E||w_\varepsilon^*(t_2, \omega) - w_\varepsilon^*(t_1, \omega)||^2 \Rightarrow 0$ in* 4) *hold uniformly with respect to all t_1, t_2, let $\psi(t)$ (cf.* 6) *and* 7)) *be estimated by a continuous functions $\psi^*(t)$: $\psi(t_2) - \psi(t_1) \leqq \psi^*(t_2 - t_1)$. Let the processes $w_\varepsilon(t, \omega)$ be homogeneous and let the equation*

$$(2') \qquad dx_\varepsilon(t, \omega) = a(t, x_\varepsilon(t, \omega), \varepsilon) \, dt + B(t, x_\varepsilon(t, \omega), \varepsilon) \, dw_\varepsilon(t, \omega)$$

for $\varepsilon = 0$ *have a constant solution* $x_0(t, \omega) = x_0(\omega)$ *for* $t \geqq t_0$ *which is asymptotically stable in average, then to every* $\eta > 0$ *there are* $\varepsilon_0 > 0$, $\sigma > 0$ *such that*

$$\sup_{\langle t_0, \infty)} E||x_\varepsilon(t, \omega) - x_0(t, \omega)||^2 < \eta \qquad \text{for } 0 \leqq \varepsilon \leqq \varepsilon_0$$

where $x_\varepsilon(t, \omega)$ *is an arbitrary solution of* (2′) *with the initial condition* $E||x_\varepsilon(t, \omega) - x_0(\omega)||^2 < \sigma$.

We can formulate sufficient conditions for the stability and the asymptotic stability in average by means of Lyapunov functions.

Let the function $F(t)$ from 1) be absolutely continuous, then there are absolutely continuous functions $F_{ij}(t)$ such that

$$E[(w_i(t_2, \omega) - w_i(t_1, \omega))\,(w_j(t_2, \omega) - w_j(t_1, \omega))] = F_{ij}(t_2) - F_{ij}(t_1)$$

where $w_i(t, \omega)$ is an i-compenent of the vector process $w(t, \omega)$. Denote by $f(t)$ and $f_{ij}(t)$ derivatives of $F(t)$ and $F_{ij}(t)$, respectively.

Theorem 3. *Let assumptions* 1) *to* 3) *be fulfilled where* $F(t)$ *is absolutely continuous and let equation* (1′) *have the solution* $x(t, \omega) \equiv 0$. *If there exists a quadratic form* $V(t, x) = \Sigma\, c_{ij}(t)\, x_i x_j$ *which fulfils the conditions that the* $c_{ij}(t)$ *have continuous second derivatives and that there are constants* $d_1 > 0$, $d_2 > 0$ *such that*

$$d_1||x||^2 \leqq V(t, x) \leqq d_2||x||^2,$$

$$W(t, x) = \frac{\partial V}{\partial t} + \sum \frac{\partial V}{\partial x_i} a_i(t, x) + \sum_{i,j,k,l} c_{ij}(t)\, B_{ik}(t, x)\, B_{jl}(t, x)\, f_{kl}(t) \leqq 0 \tag{3}$$

for almost all $t \geqq 0$, *then the solution* $x(t, 0) \equiv 0$ *is stable in average.*

Theorem 4. *Let the assumptions from Theorem* 3 *be fulfilled with* (3) *replaced by* $W(t, x) \leqq -d_3||x||^2$ *for almost all* t, $d_3 > 0$, *then* $x(t, \omega) \equiv 0$ *is asymptotically stable in average.*

The following question is of interest in the averaging theory. Under what conditions the stability of the unperturbed equation (i.e. equation (2′) for $\varepsilon = 0$) implies the stability of (2′) for small $\varepsilon > 0$ and under what conditions the existence of a periodic solution of the unperturbed equation implies the existence of such solution of (2′) for small $\varepsilon > 0$. Considering this problem we compare equation (2′) with the deterministic equation

$$\mathrm{d}y = a(t, y, 0)\, \mathrm{d}t \tag{4′}$$

with random initial values. Conditions 4) to 8) must be now reformulated:

4*) The processes $w_\varepsilon(t, \omega)$ are now defined only for $\varepsilon \in (0, \delta)$, they are processes with stochastically independent increments and there is a continuous function $F(t)$ (independent of ε) such that

$$E\|w_\varepsilon(t_2, \omega) - w_\varepsilon(t_1, \omega)\|^2 \leqq F(t_2) - F(t_1), \qquad E(w_\varepsilon(t_2, \omega) - w_\varepsilon(t_1, \omega)) = 0.$$

Assumptions 5) and 5′) are not necessary.

6*) $a(t, x, \varepsilon)$ is defined for $\varepsilon \in \langle 0, \delta \rangle$ and fulfils condition 2) where the constant K is independent of ε and

$$\int_0^t (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau), 0))\, d\tau \to 0 \qquad \text{for } \varepsilon \to 0$$

uniformly with respect to constant vectors y and $t \in \langle 0\ L \rangle$ for every $L > 0$, where $y(t)$ are solutions of (4′) with the initial conditions $y(0) = y$.

7*) $B(t, x, \varepsilon)$ is defined for $\varepsilon \in (0, \delta)$ and fulfils condition 2) where the constant K is independent of ε and

$$\int_0^t \|B(\tau, y(\tau), \varepsilon)\|^2\, dF_\varepsilon(\tau) \to 0 \qquad \text{for } \varepsilon \to 0$$

uniformly with respect to constant vectors y and $t \in \langle 0, L \rangle$ for every $L > 0$, where $y(t)$ have the same meaning as in 6*).

8) The partial derivatives $\frac{\partial a}{\partial x}, \frac{\partial B}{\partial x}$ exist and they are Lipschitz continuous in x.

The asymptotic stability in average will be replaced by exponential stability in average, too.

Definition 4. *The solutions of* (1′) *are uniformly exponentially stable in average, if they are stable in average and there exist positive constants $K > 0$ and $0 < \beta < 1$ such that*

$$E\|x^{(1)}(t, \omega) - x^{(2)}(t, \omega)\|^2 \leqq \beta E\|x^{(1)}(t_0, \omega) - x^{(2)}(t_0, \omega)\|^2 \qquad \text{for } t \geqq t_0 + K$$

for all the solutions of (1′).

Definition 5. *A process $z(t, \omega)$ is periodic with period T, if*

$$P(z(t_1, \omega) \in A_1, z(t_2, \omega) \in A_2, \ldots, z(t_s, \omega) \in A_s) =$$
$$= P(z(t_1 + kT, \omega) \in A_1, z(t_2 + kT, \omega) \in A_2, \ldots, z(t_s + kT, \omega) \in A_s)$$

for all n-dimensional Borel sets A_i, all $t_1 < t_2 < \ldots t_s$ and for all integers k.

Theorem 5. *Let the assumptions* 4*), 6*) *to* 8*) *be fulfilled, let $a(t, x, \varepsilon)$, $B(t, x, \varepsilon)$ be periodic functions in t with the period T and let $w_\varepsilon(t + h, \omega) - w_\varepsilon(t, \omega)$ be periodic processes with the same period T. If the solutions of equation* (4′) *are uniformly exponentially stable in average, then there is an $\varepsilon_0 > 0$ such that the solutions of* (2′) *are uniformly exponentially stable in average for $0 < \varepsilon \leqq \varepsilon_0$ and there exist periodic solutions $x_\varepsilon^*(t, \omega)$ of* (2′) *for $0 < \varepsilon \leqq \varepsilon_0$ and a deterministic periodic solution $y^*(t)$ of* (4′) *and*

$$\lim_{\varepsilon \to 0} \sup_{t \geqq 0} E\|x_\varepsilon^*(t, \omega) - y^*(t)\|^2 = 0$$

holds.

This Theorem has an interesting consequence for parabolic differential equations. If $a(t, x, \varepsilon)$, $B(t, x, \varepsilon)$ fulfil the assumptions of Theorem 5 and if we add some assumptions which are used in the theory of parabolic equations (e.g. $B^T B$ is positive definite for all positive ε, B^T is the transpose matrix, that B are HÖLDER continuous in t and there are $\frac{\partial a_i}{\partial x_i}, \frac{\partial B_{ij}}{\partial x_j}, \frac{\partial^2 B_{ij}}{\partial x_i\, \partial x_j}$ which are continuous and bounded), then for small $\varepsilon > 0$ the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} \frac{\partial^2 \left(\sum_k B_{ik}(t, x, \varepsilon)\, B_{jk}(t, x, \varepsilon) u\right)}{\partial x_i\, \partial x_j} - \sum_i \frac{\partial (a_i(t, x, \varepsilon)\, u)}{\partial x_i}$$

has periodic solutions with the initial values $\alpha f_0(x)$ where α is an arbitrary real number and $\int f_0(x)\, dx = 1, f_0 \geqq 0$. These solutions are relatively asymptotically stable in the sense that

$$\lim_{t \to \infty} \int_{\lambda_1}^{\mu_1} \dots \int_{\lambda_n}^{\mu_n} (u(t, x; f_1) - \alpha u(t, x; f_0))\, dx = 0$$

uniformly with respect to $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$ where $u(t, x; f_i)$ is the solution of the parabolic equation with the initial condition $u(0, x; f_i) = f_i$, if $\int |f_1(x)|\, dx < \infty$ and $\int f_1(x)\, dx = \alpha$ holds.

LITERATURE

[1] Гихман И. И.: *Дифференциальные уравнения со случайными функциями. Зимняя школа по теории вероятностей и математической статистике. Киев 1964.*