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BIFURCATION ANALYSIS OF STIMULATED BRILLOUIN SCATTERING

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1. Introduction

Our problem is motivated by the following physical effect (Stimulated Brillouin Scattering): A laser beam of a given frequency is targeted on a material sample. If the laser intensity is small then the beam penetrates without being affected. If the intensity is above a threshold then the sample acts like a mirror and reflects some energy back (Stokes' wave). This is due to stimulated pressure (acoustic) wave in the sample. The frequences of all three waves (i.e. laser and Stokes and pressure) are coupled.

Let us accept that Stimulated Brillouin Scattering can be modelled by the following initial value problem: Find complex-valued functions E_L, E_S, p (the slowly varying amplitudes of laser and Stokes and pressure waves) of time $t \geqq 0$ and one spatial variable $0 \leqq x \leqq \ell$ such that

$$\dot{E}_L = E_L' - iE_S p \ , \quad \dot{E}_S = - E_S' - iE_L\bar{p} \ , \quad c\dot{p} = p' - p - iE_L\bar{E}_S \tag{1.1}$$

(Notation: $\dot{E} = \frac{\partial}{\partial t} E$, $E' = \frac{\partial}{\partial x} E$, $\bar{E}$ is complex conjugate, $i = (-1)^{1/2}$, c is a positive constant) with boundary condition

$$p(\ell,t) = E_S(0,t) = 0 \ , \quad E_L(\ell,t) = ae^{i\psi} \tag{1.2}$$

for $t \geqq 0$; a and ψ are real parameters. At $t = 0$, an initial condition (compatible with (1.2)) is prescribed.

1.3 REMARK. The sample occupies the interval $0 \leqq x \leqq \ell$. Laser light of intensity a^2 is focused at the point $x = \ell$ and propagates in negative direction of the x-axis. Note that functions $E_L \equiv ae^{i\psi}$, $E_S \equiv p \equiv 0$ are a steady state solution to (1.1), (1.2). This *trivial solution* corresponds to the situation when no stimulation of pressure waves occurs which is expected if a^2 is less then the threshold. Physical significance of the model (1.1), (1.2) is discussed e.g. in [1].

1.4 OBSERVATION. If E_L , E_S , p solve (1.1) then, for each real constant γ , the functions $E_L e^{i\gamma}$, E_S , $pe^{i\gamma}$ solve (1.1), too. It implies that we can *assume* $\psi = 0$ *and* $a \geq 0$ without loss of generality.

Our aim is bifurcation analysis of trivial solution to steady state problem (1.1), (1.2) with respect to variations of parameter a . We mention dynamical stability of bifurcated solutions, too.

2. Covariance of the equations governing the steady state

Let us introduce the operator F of the steady state: Homogenising the boundary condition (1.2) by substitution $E_L := E_L + a$, we define $F = F(\mathcal{U},a)$ at $\mathcal{U} = (E_L, E_S, p)$ and at $a \in \mathbb{R}_1$ as follows:

$$F(\mathcal{U},a) = (E_L' - iE_S p, \ - E_S' - iE_L\bar{p} - ia\bar{p}, \ p' - p - ia\bar{E}_S - iE_L\bar{E}_S) .$$

Then F acts (e.g.) on the *real* linear space $X = \{\mathcal{U} = (E_L,E_S,p) : E_L, E_S, p$ are complex-valued functions of x , continuously differentiable on $0 \leq x \leq \ell$, satisfying boundary condition $E_L(\ell) = E_S(0) = p(\ell) = 0\}$. The range of $F(.,a)$ is in the real linear space $Y = \{\mathcal{U} = (E_L,E_S,p) : E_L, E_S, p$ are complex-valued, continuous functions of x , $0 \leq x \leq \ell\}$. In the usual topology, there is compact imbedding of X into Y .

Let us define suitable linear transformations on Y : If $\mathcal{U} = (E_L, E_S, p) \in Y$ then

$$M_\beta \mathcal{U} = (E_L, \ e^{i\beta}E_S, \ e^{-i\beta}p) \quad \text{for each} \quad \beta \in \mathbb{R}_1 ,$$

$$T_j \mathcal{U} = (\bar{E}_L, \ (-1)^j\bar{E}_S, \ (-1)^{j+1}\,\bar{p}) \quad \text{for} \quad j = 1,2 .$$

Let $\{M_\beta,T_1,T_2\}$ denote the group generated by M_β , T_1 , T_2 $(\beta \in \mathbb{R}_1)$. The covariance property of the steady state operator follows from

2.1 LEMMA. *If* $\mathcal{U} \in X$ *then* $M_\beta F(\mathcal{U},a) = F(M_\beta\mathcal{U},a)$ *and* $T_j F(\mathcal{U},a) = F(T_j\mathcal{U},a)$ *for each* $\beta \in \mathbb{R}_1$ *and* $j = 1,2$.

Proof. This can be done by a straightforward calculation.

As a direct consequence, we obtain

2.2 PROPOSITION. *If* $\Gamma \in \{M_\beta,T_1,T_2\}$ *then* $\Gamma F(\mathcal{U},a) = F(\Gamma\mathcal{U},a)$ *for each* $\mathcal{U} \in X$.

3. Bifurcation analysis

We resume the steady state problem: Given a value of $a > 0$, find $U \in X$ such that $F(U,a) = 0$. Obviously, $U^0 = 0$ is trivial solution for each value of a.

Let $L(a) : X \to Y$ be Fréchet derivative of the operator $F(.,a) : X \to Y$ at U^0. If there is a bifurcation from U^0 at $a \in \mathbb{R}_1$ then the kernel *Ker* $L(a)$ is nontrivial. Direct calculations yield

3.1 LEMMA. *If* $a \geq 0$ *then Ker* $L(a)$ *is nontrivial if and only if* $a \in K$, *where* $K = \{a \geq 0 : 4a^2 - 1 \geq 0 , \ \mathrm{tg}\, \frac{\omega \ell}{2} = - \omega , \ \omega = (4a^2 - 1)^{\frac{1}{2}}\}$. *The set* K *can be arranged as an increasing sequence* $\{a_j\}_{j=1}^{\infty}$, $0 < a_1 < a_j < a_{j+1}$, $\lim\limits_{j \to +\infty} a_j = + \infty$.

3.2 LEMMA. *If* $a_j \in K$ *then Ker* $L(a_j)$ *is spanned by vectors* $\xi_1 = (0, v_0, -iw_0)$, $\xi_2 = (0, -iv_0, w_0)$ *from* X, *where* v_0 *and* w_0 *are the following real functions of* x : $v_0 = e^{x/2} \sin \frac{\omega x}{2}$, $w_0 = (-1)^{j+1} e^{x/2} \sin \frac{\omega(\ell - x)}{2}$; $\omega = (4a_j^2 - 1)^{1/2}$.

Let us make some remarks on bifurcation equation: By virtue of Fredholm alternative, space Y can be decomposed as a direct sum of kernel and range of the operator $L(a_j) : X \to Y$. Thus each solution U to $F(U,a)$ can be written as $U = \sum y_i \xi_i + U^{\perp}$ where y_i's are real coordinates, ξ_i's span the kernel and $U^{\perp}$ belongs to the range. According to Liapunov-Schmidt reduction (see e.g. [3]), if U and a are sufficiently close to U^0 and a_j respectively then U can be *identified* with coordinates y_1, y_2 of the projection into *Ker* $L(a_j)$. The coordinates satisfy *bifurcation equation* $H(y_1, y_2, a - a_j) = 0$, where H is germ of a mapping $H : \mathbb{R}_2 \times \mathbb{R}_1 \to \mathbb{R}_2$.

Following standard routines, Taylor expansion of H can be found. In our particular case, we have calculated that

$$H(y_1, y_2, a - a_j) = p_j (a - a_j) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - q_j (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + s_j (a - a_j)^2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \text{terms of the 4 th order} \tag{3.3}$$

where p_j, q_j, s_j are real constants, namely $p_j = \ell + 2 \cos^2 \frac{\omega \ell}{2}$,

$q_j = (-1)^j \cos \frac{\omega \ell}{2} \int_0^{\ell} (v_0^2(\ell) - v_0^2)(v_0^2 + w_0^2)\, dx$; for ω, v_0, w_0 see 3.2.

Both p_j and q_j are positive.

3.4 OBSERVATION. Group $\{M_\beta, T_1, T_2\}$ leaves the kernel *Ker* $L(a_j)$ invariant. The matrix representation of $\{M_\beta, T_1, T_2\}$ on *Ker* $L(a_j)$ is $O(2)$ group of 2×2 orthogonal matrices. Following Sattinger [2], the bifurcation equation is covariant under $O(2)$ symmetry group.

3.5 THEOREM. *The bifurcation equation* $H(y_1, y_2, a - a_k) = 0$ *is* $O(2)$*-equivalent to*

$$\left[a - a_k - (y_1^2 + y_2^2)\right] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

(*for the notion of* $O(2)$*-equivalence, see* [4]).

Proof. We just quote [4], Lemma 5.18 (our points 3.3 and 3.4 verify the assumptions).

We conclude that each $a_j \in K$ is a point of supercritical bifurcation. Bifurcation diagrams are 2 - D manifolds (symmetric, in a sense, about a-axis).

4. Stability

Let us resume the initial value problem (1.1), (1.2) where we substitute $E_L := E_L + a$: Given $a \geq 0$ and $U^{in} \in X$, find $U = (E_L, E_S, p)$ such that $\dot{U} = F(U,a)$ for $t > 0$ and $U = U^{in}$ at $t = 0$. Asymptotic stability (as $t \to +\infty$) of the above problem is under question, assuming that U^{in} is a steady state solution being subjected to a small perturbation. In [5] we have tackled this question by making use of Principle of Linearised Stability.

We just quote our results (which are not surprising anyway) : Trivial solution U^0 is stable as $0 \leq a < a_1$ (i.e. up to the first bifurcation point). Beyond this point, it looses stability. Nontrivial steady state solutions which emanate from bifurcation points $a_2, a_3, \ldots$ are unstable. On the other hand, the branch emanating from a_1 is stable.

The above facts are illustrated by the following numerical experiments.

EXAMPLE 1. Data: $\ell = c = 1$, $a = 1$ (i.e. $a < a_1$) ; U^{in} is just the trivial solution U^0 being perturbed by Gaussian "noise" (δ-correlated, dispersion = 1 , mean value = 0). Time interval: $0 \leq t \leq 0.3$.

Results are presented in Figure 1 : At each time, the (numerical) solution U is projected onto *Ker* $L(a_1)$ and then on y_1, y_2-plane.

Point S is projection of U^{in} . It is apparent that solution creeps towards the origin, i.e. U^0 . The second graph indicates velocity of the motion in time.

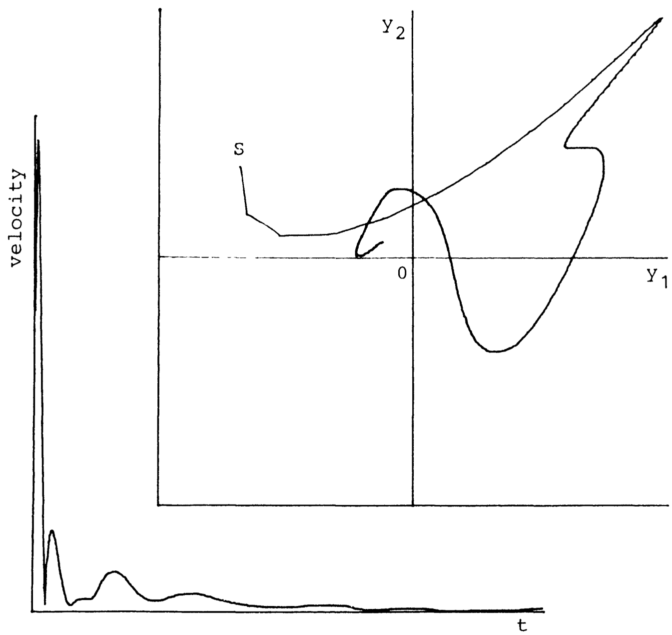

Figure 1

<u>EXAMPLE 2</u>. Data: $\ell = c = 1$, $a = 5$ (i.e. $a_2 < a < a_3$) ; U^{in} is a (numerical) steady state solution on the 2^{nd} branch which is randomly perturbed as above. Time interval: $0 \leq t \leq 10$.

Legend to Figure 2 : U is projected onto $Ker\ L(a_1)$ again. S is the position of projected U^{in} . Note that projection of all steady solutions on the first branch at $a = 5$ would be a circle centred at origin O and passing through C . We observe U oscillating around a point on this circle for large t .

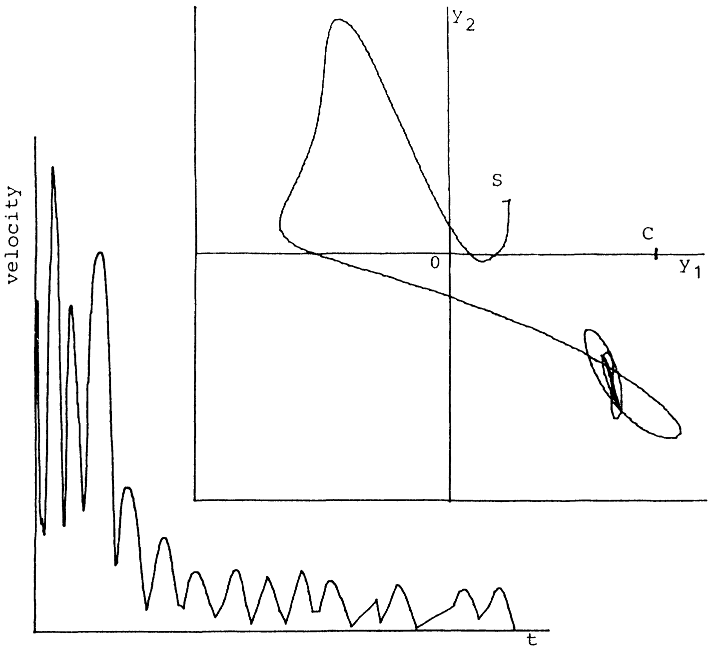

Figure 2

R e f e r e n c e s

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