Jerrold W. Bebernes

A description of blow-up for the solid fuel ignition model

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [193]--196.

Persistent URL: http://dml.cz/dmlcz/700189

A DESCRIPTION OF BLOW-UP FOR THE SOLID FUEL IGNITION MODEL

J. W. BEBERNES
Department of Mathematics, University Colorado
Boulder, CO 80309, U.S.A.

The nondimensional ignition model for a supercritical high activation energy thermal explosion of a solid fuel in a bounded container Ω can be described by

(1) $u_t - \Delta u = e^u$

(2) $u(x,0) = \phi(x) \geq 0,\ x \in \Omega,\ u(x,t) = 0,\ x \in \partial\Omega,\ t > 0$

where $\Omega = \{x \in \mathbb{R}^n: |x| \leq R\}$ and ϕ is radially decreasing, i.e., $\phi(x) \geq \phi(y) \geq 0$ whenever $|x| \leq |y| \leq R$ and $\Delta\phi + e^{\phi} \geq 0$ on Ω.

Assume $R > 0$ is such that the radially symmetric solution $u(x,t)$ blows up in finite time $T > 0$. Then by the maximum principle $u(\cdot,t)$ is radially decreasing for $t \in [0,T)$ and $u_t(x,t) \geq 0$ for all $(x,t) \in \Gamma = \Omega \times [0,T)$.

Friedman and McLeod [4] recently proved that blow-up occurs only at the origin $x = 0$ and in addition that $u(x,t)$ satisfies the following estimates: I) $u(x,t) \leq -\frac{2}{\alpha}\ \ell n|x| + c$ for all $\alpha < 1$ and $(x,t) \in \Gamma$; II) there exists $\bar{t} < T$ such that $|\nabla u(x,t)| \leq 2e^{u(0,t)/2}$, $t \in [\bar{t},T)$, $|x| \leq R$; III) there exists $\delta > 0$ such that $u_t(x,t) \geq \delta e^{u(x,t)}$, $t \in [\frac{T}{2}, T)$, $x \in [-\frac{R}{2}, \frac{R}{2}]$; and iv) $-\ell n(T-t) \leq u(0,t) \leq -\ell n(T-t) - \ell n\delta$, $t \in [\frac{T}{2}, T)$, $\delta > 0$.

Since $u(x,t)$ is radially symmetric, the initial boundary value problem (1)-(2) can be reduced to a problem in one spatial dimension. Let $D = \{(r,t): 0 \leq t \leq T,\ 0 \leq r \leq R\}$. Then if $r = |x|$, $v(r,t) = u(x,t)$ satisfies:

(3) $v_t = v_{rr} + \frac{n-1}{r} v_r + e^v$

(4) $v(r,0) = \phi(r),\ v_r(0) = 0,\ v(R,t) = 0.$

To study the asymptotic behavior of v as $t \to T$, consider the following change of variables: $\tau = -\ell n(T-t)$, $\eta = r(T-t)^{-1/2}$, $\theta = v + \ell n(T-t) = v - \tau$ whose inverse is $t = T - e^{-\tau}$, $r = \eta e^{-\tau/2}$, $v = \theta - \ell n(T-\tau)$. The domain D transforms to $D' = \{(\eta,\tau): 0 \leq \eta \leq Re^{\tau/2},\ \tau \geq -\ell nT\}$ and $\theta(\eta,\tau) = v - \tau$ solves

(5) $\theta_t = \theta_{\eta\eta} + \left(\frac{n-1}{\eta} - \frac{\eta}{2}\right)\theta_\eta + e^\theta - 1$

(6) $\theta(\eta, -\ell n T) = \phi(\eta T^{1/2}) + \ell n T$

$\theta_\eta(0,\tau) = 0, \ \theta(Re^{\tau/2},\tau) = -\tau$

The following theorem is similar to a result proven by Giga-Kohn [5].

Theorem 1. As $\tau \to +\infty$, the solution $\theta(\eta,\tau)$ tends uniformly to a function $y(\eta)$ on compact subsets of $\mathbb{R}^+$ where $y(\eta)$ is a solution of the problem:

(7) $y'' + \left(\frac{n-1}{\eta} - \frac{\eta}{2}\right)y' + e^y - 1 = 0$

(8) $y'(0) = 0, \ y(0) = \alpha \geq 0$

which is globally Lipschitz continuous and nonincreasing in η.

Thus, to describe how the blow-up occurs at (T,0) for (1)-(2), we need to analyze the solutions of the steady-state equation (7)-(8) which are globally Lipschitz and are nonincreasing on $[0,\infty)$.

Theorem 2. For $n = 1$ or 2, the only solution of (7)-(8) which is globally Lipschitz continuous and nonincreasing in η is $y(\eta) \equiv 0$.

Proof. For n = 1, this result was first proven by Bebernes-Troy [2]. The following proof is essentially due to D.Eberly. For n > 2, the proof fails. Let

$g(\eta) = \frac{\eta}{2} y'(\eta) + 1$

and $h(\eta) = y''(\eta) + \frac{n-1}{\eta} y'(\eta)$

where $y(\eta)$ is a solution of (7) - (8).

Then $g(\eta)$ satisfies

(9) $\begin{cases} g'' + (\frac{n-1}{\eta} - \frac{\eta}{2})g' + (e^y - 1)g = 0 \\ g(0) = 1, \quad g'(0) = 0 \end{cases}$

and $h(\eta)$ satisfies

(10) $\begin{cases} h'' + (\frac{n-1}{\eta} - \frac{\eta}{2})h' + (e^y - 1)h \leq 0 \\ h(0) = 1 - e^\alpha, \ h'(0) = 0. \end{cases}$

It is clear that $g(\eta) > 0$ on $I = [0,x_0)$ where $x_0 \in (0,\infty]$.

Set $W(\eta) = gh' - g'h$, then $W(\eta)$ satisfies

(11) $\begin{cases} W' + (\frac{n-1}{\eta} - \frac{\eta}{2})W = -e^y(y')^2 g(\eta) < 0 \\ W(0) = 0 \end{cases}$

on I. This implies $W(\eta) \le 0$ on I and hence $h(\eta)/g(\eta) \le h(0)/g(0) = 1 - e^{\alpha}$ on I. Thus, we have

(12) $h(\eta) \le (1 - e^{\alpha})g(\eta)$ on I.

We now must consider two cases. We assume now that $n = 1$ or 2.

a) If $x_0 < \infty$, then $g(x_0) = 0$ and $g' - \frac{\eta}{2}g = -\frac{\eta}{2}e^{y} < 0$ implies $g(\eta) < 0$ for all $\eta > \eta_0$. Thus $(\eta y')' = \eta g(\eta) - \eta e^{y} < 0$ and $y(\eta)$ is not globally Lipschitz on $[0,\infty)$.

b) If $x_0 = +\infty$ and $g(\eta) > \varepsilon > 0$ for all $\eta \ge 0$, then $(\eta y')' < 0$ by (12) and again $y''(\eta) < 0$. If $\liminf g(\eta) = 0$ as $\eta \to \infty$ with $g(\eta) > 0$, we observe that (11) can be solved for $h(\eta)$ to give

$$(13)\quad h(\eta) = (1 - e^{\alpha})g(\eta) - $$

$$- g(\eta)\int_0^{\eta} \frac{1}{g^2(s)} \frac{e^{s^2/4}}{s} \left(\int_0^{s} u e^{-u^2/4}\, e^{y}(y')^2 g(u)\,du\right)ds$$

By analyzing (13), we can show that $h(\eta) \to -\infty$ as $\eta \to +\infty$. Once again we have that $y''(\eta) < 0$ for η large and $y(\eta)$ cannot be globally Lipschitz on $[0,\infty)$. This completes the proof in dimensions 1 and 2.

As an immediate consequence of theorems 1 and 2, we have

Theorem 3. Let $n = 1$ or 2. As $t \to T^-$, $v(r,t) - \ln(T - t)^{-1} \to 0$ uniformly on $0 \le r \le c(T - t)^{1/2}$.

These results will appear in [3].

Several open questions remain. What can be said for $n \ge 3$? What happens outside the parabolic domain $r \le c(T-t)^{1/2}$ as $t \to T^-$?

References

[1] J.Bebernes and D.Kassoy, *A mathematical analysis of blowup for thermal reactions - the spatially monhogeneous case*, SIAM J. Appl. Math. 40 (1981), 476-484.

[2] J.Bebernes and W.Troy, *Nonexistence for the Kassoy problem*, SIAM J. Math. Analysis, submitted.

[3] J.Bebernes, A.Bressan and D.Eberly, *A description of blow-up for the solid fuel ignition model*, submitted.

[4] A.Friedman and B.McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. 34 (1985), 425-447.

[5] Y.Giga and R.Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. 38 (1985), 297-320.

[6] D. Kassoy and J.Poland, *The thermal explosion confined by a constant temperature boundary:I. The induction period solution*, SIAM J. Appl. Math. 39 (1980), 412-430.