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CRITICAL POINT THEORY AND NONLINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The variational approach to boundary value problems for differential equations consists in writing the problem, whenever it is possible, as an abstract equation of the form

(1) $$\Phi(u) = 0$$

where $\Phi : E \to E^*$ is of the form $\Phi = \varphi'$, with φ' the Gâteaux derivative of a real function φ defined on a Banach space E. In this way the search of solutions for (1) is equivalent to the determination of the critical points of Φ, i.e. of the zeros of φ'. Such a viewpoint can be traced at least to Fermat, with his minimal type principle to find the law of refraction for the light.

Since Fermat also, we know that the points at which φ achieves its extremums are critical points of φ. Thus, any way which succeeds in proving, directly, that φ has a maximum or a minimum provides a way of proving the existence of a solution of (1). This is the so-called direct method of the calculus of variations which goes back to Gauss, Kelvin, Dirichlet, Hilbert, Tonelli and others. More recent work deals with proving the existence of critical points at which φ does not achieve an extremum (saddle points). This paper surveys some of the recent work in this direction. A systematic exposition of many aspects of the variational approach to boundary--value problems for ordinary differential equations will be given in [11].

For definiteness, we shall consider a system of ordinary differential equations of the form

(2_α) $$u'' + \alpha u = \nabla F(x,u) \qquad (\nabla = D_u)$$

on a compact interval $I = [a.b]$, submitted to homogeneous boundary conditions, say, of Dirichlet, Neumann or periodic type. For simplicity, we assume here that F and ∇F are continuous on $I \times \mathbf{R}^N$. We could as well consider elliptic partial differential equations. It is well known that the spectrum of $-d^2/dt^2$ submitted on I to the above boundary conditions has the form

$$(0\leq)\ \lambda_1 < \lambda_2 < \ldots .$$

Moreover, (2) is the Euler-Lagrange equation associated to the functional

$$\varphi : H \to \mathbf{R},\ u \mapsto Q_\alpha(u) + \int_I F(.,u(.))$$

where

$$Q_\alpha(u) = \int_I (1/2)(|u'|^2 - \alpha|u|^2),$$

and $H = H_0^1(I;\mathbf{R}^N)$, $H^1(I;\mathbf{R}^N)$ or $H_\#^1(I,\mathbf{R}^N) = \{u \in H^1(I,\mathbf{R}^N) : u(a) = u(b)\}$ with their usual norm denoted by $\|.\|$. Solving (2) with one of the above boundary condition is thus equivalent to finding a *critical point* of φ on H, i.e. a point $u \in H$ such that

$$(3) \qquad \varphi'(u) = 0.$$

If $c = \varphi(u)$ with u a critical point, c is called a *critical value* for φ.

The simplest situation for (3) to hold is when φ has a global minimum (which requires of course φ to be bounded from below).

Since Hammerstein [6] in 1930 (in the Dirichlet case) we know that φ will have a global minimum whenever

$$(4) \qquad \alpha < \lambda_1$$

and

$$(5) \qquad F(x,u) \geq -(\beta/2)|u|^2 - \gamma(x)$$

for some $\beta < \lambda_1 - \alpha$, $\gamma \in L^1(I)$ and all $(x,u) \in I \times \mathbf{R}^N$. In fact, φ is coercive ($\varphi(u) \to +\infty$ for $\|u\| \to \infty$) because, by (4) and (5), φ is bounded from below by a coercive quadratic form. Moreover φ is weakly lower semi-continuous so that φ has a global minimum by a classical result. We shall discuss now situations where (4) and (5) do not hold.

2. THE CASE OF $\alpha = \lambda_1$ AND $\int_I F$ COERCIVE ON THE KERNEL

The situation is already more complicated when $\alpha = \lambda_1$ (*resonance at the lowest eigenvalue*) and condition (5) is no more sufficient for the existence of a critical points as shown by a linear equation violating the Fredholm alternative condition. To motivate the introduction of a new-sufficient condition, let us first consider the case where ∇F is bounded.

a) The case where ∇F is bounded

Writing $u(x) = \bar{u}(x) + \tilde{u}(x)$ with $\bar{u} \in \bar{H}_1$ the eigenspace of λ_1 and $\tilde{u} \in \tilde{H}_1 = H_1^{\perp}$, we have $\varphi(u) = Q_{\lambda_1}(\tilde{u}) + \int_I [F(.,\bar{u}(.)) + F(.,u(.)) -$

$$- F(.,\bar{u}(.))] \geq Q_{\lambda_1}(\tilde{u}) + \int_I F(.,\bar{u}(.)) -$$
$$- M\|\tilde{u}\|_{L^2} \geq c_1\|\tilde{u}\|^2 - c_2\|\tilde{u}\| + \int_I F(.,\bar{u}(.)),$$

where M is an upper bound for $|\nabla F|$ on $I \times \mathbf{R}^N$, and we shall recover coercivity for φ if we assume that

$$(6) \qquad \int_I F(.,v(.)) \to +\infty \quad \text{as } \|v\| \to \infty \text{ in } \bar{H}_1$$

(*coercivity of the averaged* F *on the kernel*). Such a condition was first introduced by Ahmad, Lazer and Paul [1] and it generalizes the classical Landesman-Lazer conditions. As φ is again w.l.s.c., the existence of a minimum is insured.

b) The case where F is convex

The boundedness of ∇F can be replaced by the convexity of $F(x,.)$ for each $x \in I$. In this case, if (6) also holds, there exists $\bar{u}_0 \in \bar{H}_1$ such that

$$(7) \qquad \int_I \nabla F(.,\bar{u}_0(.))\bar{v} = 0 \quad \text{for all } \bar{v} \in \bar{H}_1.$$

Moreover, by convexity and using (7) we have

$$\varphi(u) \geq Q_{\lambda_1}(\tilde{u}) + \int_I [F(.,\bar{u}_0(.)) + (\nabla F(.,\bar{u}_0(.)),u - \bar{u}_0)]$$
$$(8) \qquad = Q_{\lambda_1}(\tilde{u}) + \int_I F(.,\bar{u}_0(.)) + \int_I (\nabla F(.,\bar{u}_0(.)),\tilde{u})$$
$$\geq c_1\|\tilde{u}\|^2 - c_2\|\tilde{u}\| - c_3$$

Thus each minimizing sequence (u_k) for φ has $(\tilde{u}_k)$ bounded in the norms $\|.\|$ and $\|.\|_{L^\infty}$. On the other hand, by convexity again

$$F(x,\bar{u}_k/2) \leq (1/2)F(x,u_k) + (1/2)F(x,-\tilde{u}_k)$$

and hence,

$$\varphi(u_k) \geq 2\int_I F(.,\bar{u}_k/2) - \int_I F(.,-\tilde{u}_k) \geq$$
$$\geq 2\int_I F(.,\bar{u}_k/2) - c_4,$$

which, by (6), implies that $(\bar{u}_k)$ is bounded and φ has a minimum.

Let us remark that when $F(x,.)$ is strictly convex for each $x \in I$ and $\alpha = \lambda_1$, it can be shown that (6) is necessary and sufficient for the existence of a solution [10].

c) <u>The case where $-\int_I F$ is coercive on the kernel</u>

Let us assume now that

$$\int_I F(.,N(.)) \to -\infty \quad \text{as} \quad \|v\| \to \infty \quad \text{in } \bar{H}_1. \tag{9}$$

As this situation only holds in trivial situations when $F(x,.)$ is convex, let us assume again that ∇F is bounded. By (9), we have

$$\varphi(v) = \int_I F(.,v(.)) \to -\infty \quad \text{as} \quad \|v\| \to \infty$$

in $\bar{H}_1$, so that φ is no more bounded from below and has no global minimum. On the other hand, on $\tilde{H}_1$,

$$\varphi(w) = Q_{\lambda_1}(w) + \int_I [F(.,0) + (F(.,w(.)) - F(.,0))] \geq$$
$$\geq c_1\|w\|^2 - c_2\|w\| - c_3$$

and hence $\varphi|_{\tilde{H}_1}$ is bounded from below (even coercive). Consequently, there exists $R > 0$ such that

$$\sup_{\bar{H}_1 \cap \partial B(R)} \varphi < \inf_{\tilde{H}_1} \varphi$$

This suggests the use of the following *saddle type* or *minimax theorem* of Rabinowitz [15], introduced to give a variational proof of the Ahmad-Lazer-Paul results [1].

LEMMA 1. *Let* E *be a Banach space and* $\varphi \in C^1(B,\mathbb{R})$. *Assume that there exists a decomposition* $E = E_1 \oplus E_2$ *with* $\dim E_1 < \infty$ *and* $R > 0$ *such that*

$$\sup_{E_1 \cap \partial B(R)} \varphi < \inf_{E_2} \varphi ,$$

Let

$$\Sigma = \{\sigma \in C(E,E) \mid \sigma(u) = u \;\; on \;\; \partial B(R)\}$$

and

$$c = \inf_{\sigma\in\Sigma} \max_{s\in B(R)\cap E_1} \varphi(\sigma(s)) \qquad (\geq \inf_{E_2} \varphi) \tag{10}$$

Assume that if there is a (u_k) *such that* $\varphi(u_k) \to c$ *and* $\varphi'(u_k) \to 0$, *then* c *is a critical value. (Palais-Smale type condition* PS^* *at* c). *Then* φ *has a critical point with critical value* c.

This theorem can be proved by deformation techniques [12] or Ekeland's variational lemma [4].

In the above case with $E = H$, $E_1 = \overline{H}_1$, $E_2 = \tilde{H}_1$, the PS^*-condition holds for each c and φ has a critical point.

The above results are summarized in the following.

THEOREM 1. *Assume that*

$$\int_I F(.,v(.)) \to +\infty \quad as \;\; \|v\| \to \infty \;\; in \;\; \overline{H}_1$$

(the eigenspace of λ_1*) and that either* ∇F *is bounded or* F *is convex in* u. *Then* (2_{λ_1}) *with the suitable boundary conditions has at least a solution which minimizes* φ. *Assume that*

$$\int_I F(.,v(.)) \to -\infty \quad as \;\; \|v\| \to \infty \;\; in \;\; \overline{H}_1$$

and that ∇F *is bounded. Then* (2_{λ_1}) *with the suitable boundary conditions has at least a solution* u *with* $\varphi(u) = c$ *given by (10) with* $E_1 = \overline{H}_1$ *the eigenspace of* λ_1.

3. THE CASE OF $\alpha = \lambda_1$ AND F PERIODIC

An interesting situation in which (6) does not hold occurs when

$$F(x, u + T_i e_i) = F(x,u) \qquad (1 \le i \le N)$$

for all $x \in I$, $u \in \mathbf{R}^N$ and some $T_i > 0$. $(1 \le i \le N)$.
This implies that F and ∇F are bounded on $I \times \mathbf{R}^N$. Therefore

$$\begin{aligned} \varphi(u) &= Q_{\lambda_1}(\tilde{u}) + \int_I F(.,u(.)) \\ &\ge c_1 \|\tilde{u}\|^2 - c_2 , \end{aligned} \tag{11}$$

so φ is bounded from below and any minimizing sequence (u_k) is such that $(\tilde{u}_k)$ is bounded in the norms $\|.\|$ and $\|.\|_{L^\infty}$.

a) The case of Neumann or periodic boundary conditions

Then, $\lambda_1 = 0$ and $\overline{H}_1 \approx \mathbf{R}^N$ is the space of constant mappings from $[a,b]$ into $\mathbf{R}^N$. Moreover,

$$\varphi(u + T_i e_i) = \varphi(u) \qquad (1 \le i \le N) \tag{12}$$

for all $u \in H$, so that any minimizing sequence can be supposed, without loss of generality, such that

$$|\overline{u}_k| \le \left(\sum_{i=1}^{N} T_i^2\right)^{1/2}.$$

Thus φ has a bounded minimizing sequence and hence a minimum. This result is due to Willem [18] and (independently and in special cases) Hamel [5] and Dancer [3]. The existence of a second solution was proved by Mawhin-Willem [8,9] using the mountain pass lemma, a variant of Lemma 1. Their approach was extended to systems of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}}(u,\dot{u}) - \frac{\partial L}{\partial u}(u,\dot{u}) = 0$$

by Capozzi, Fortunato and Salvatore [2]. See also Pucci-Serrin [13, 14] for abstract critical point theorems motivated by this situation.

b) The case of Dirichlet boundary conditions

The Dirichlet case strongly differs from the other ones because

$\lambda_1 = \frac{\pi^2}{(b-a)^2} > 0$ and $\overline{H}_1 = \text{span}(\sin \frac{\pi x}{b-a})$ which imply that we loose the periodicity property (12) of φ. The problem has been studied by Ward [17] for N = 1 and

$$(13) \qquad F(x,u) = G(u + E(x))$$

where G is continuous and T-periodic and $E : I \to \mathbf{R}$ is continuous. Indeed, Ward considered explicitely the problem

$$v'' + \lambda_1 v = g(v) + e(t)$$
$$v(a) = v(b) = 0$$

when $g(v + T) = g(v)$, $\int_0^T g = 0$ and $e \in \tilde{H}_1$, which reduces to the above case by a trivial change of variables.

A possible way of approach, slightly different from Ward's one, makes use of the following lemma which can be proved by a deformation technique or Ekeland variational lemma.

LEMMA 2. *Let E be a Banach space and $\varphi \in C^1(E,\mathbf{R})$ be bounded from below and satisfy PS^* at $c = \inf \varphi$. Then φ has a minimum.*

Using an extension of the Riemann-Lebesgue lemma, one can prove that φ associated to F given in (13)satisfies the PS^*-condition at each $b \neq 0$ and that $\varphi|_{\tilde{H}_1}$ satisfies PS^* at each $b \in \mathbf{R}$. Thus the existence of a critical point is insured by Lemma 2 except when

$$0 = \inf_H \varphi < \inf_{\tilde{H}_1} \varphi$$

The above mentioned Riemann-Lebesgue type lemma also implies that, on $\overline{H}_1$, $\varphi(v) \to 0$ as $\|v\| \to \infty$. Thus, there exists some $R > 0$ such that

$$\max_{\overline{H}_1 \cap \partial B(R)} \varphi < \inf_{\tilde{H}_1} \varphi$$

and then c given by the Rabinowitz lemma is greater or equal to $\inf \varphi$ and hence nonzero. Consequently, this c is a critical value for φ.

The above results can be summarized in the following.

THEOREM 2. *Assume that*

$$F(x,u + T_i e_i) = F(x,u) \qquad (1 \le i \le N)$$

with N = 1 *and* F *of the form* (13) *in the Dirichlet case. Then* (2_{λ_1}) *with the suitable boundary condition has at least one solution.*

4. THE CASE OF $\lambda_{i-1} < \alpha \le \lambda$ $(i \ge 2)$

In this case, φ is neither bounded from below nor from above, as $Q_\alpha(v) \to -\infty$ on $\overline{H}_{i-1}$ = span of eigenfunctions of $\lambda_1,\ldots,\lambda_{i-1}$ and $Q_\alpha(v) \to +\infty$ on $\tilde{H}_{i+1}$ = span of eigenfunctions of $\lambda_{i+1},\ldots$.

a) The case where ∇F is bounded

Then one can use the Rabinowitz Lemma in a way similar to the case where $\alpha = \lambda_1$ and $\int_I F(.,v(.)) \to -\infty$ as $\|v\| \to \infty$ if the extra condition

(14) $\int_I F(.,v(.))dx \to +\infty$ or $-\infty$ as $\|v\| \to \infty$ in the eigenspace of λ_i

holds when $\alpha = \lambda_i$. One choose in this case $E_1 = \overline{H}_i$, $E_2 = \tilde{H}_i$ or $E_1 = \overline{H}_{i+1}$, $E_2 = \tilde{H}_{i+1}$ according to the sign of ∞ in (14). Under these conditions (2_α) has at least one solution. This is essentially a result of Ahmad-Lazer-Paul [1] and Rabinowitz [15].

b) The case where F is convex

Then, sharper results can be obtained without boundedness assumption of ∇F through the use of the Clarke-Ekeland dual least action principle which reduce the study of the critical points of φ to that of an associate dual function ψ involving the (possibly generalized) inverse of $\frac{d^2}{dt^2} + \lambda_i I$ and the Legendre-Fenchel transform of $F(x,.)$. Under reasonable conditions on F, ψ is bounded from below and, in this way the existence of a solution is in particular insured when

$$\limsup_{|u|\to\infty} \frac{F(x,u)}{|u|^2} \le \beta < \frac{\lambda_{i+1} - \lambda_i}{2} \quad \text{(unif. in } x \in I)$$

and (if $\lambda_i = \alpha$),

$$\int_I F(x,f(x))dx \to +\infty \text{ as } \|v\| \to \infty \text{ in the eigenspace of } \lambda_i.$$

See [10] for general results in this direction.

c) The case where F is periodic and $\alpha = \lambda_i$

Results are known only when N = 1 and F has the form (13). The proof, due to Lupo and Solimini [16,7] is such more delicate because the PS* is not satisfied at c = 0. This requires, in addition to the classical Rabinowitz saddle point theorem, other saddle point theorems of the same type and some topological arguments (together with the Riemann-Lebesgue-type lemma mentioned above).

The above results can be summarized in the following

THEOREM 3. *Assume that* $\lambda_{i-1} < \alpha \le \lambda_i$ $(i \ge 2)$ *and that one of the following conditions holds:*

i) ∇F *is bounded and, whenever* $\alpha = \lambda_i$,

$\int_I F(.,v(.)) \to +\infty$ *or* $-\infty$ *as* $\|v\| \to \infty$ *in the eigenspace of* λ_i

ii) $F(x,.)$ *is convex,* $\limsup_{|u| \to \infty} \frac{F(x,u)}{|u|^2} \le \beta < \frac{\lambda_{i+1} - \lambda_i}{2}$ *(unif. in* $x \in I$),

and, whenever $\alpha = \lambda_i$,

(14) $\int_I F(.,v(.)) \to +\infty$ *as* $\|v\| \to \infty$ *in the eigenspace of* λ_i

iii) $\alpha = \lambda_i$, N = 1 *and* F *has the form (13) with* G *T-periodic.*

Then the problem (2_α) *with any of the boundary conditions has at least one solution.*

One can show that (14) is necessary and sufficient when F(x,.) is strictly convex [10].

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