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SOME NEW CONVERGENCE RESULTS IN FINITE ELEMENT THEORIES FOR ELLIPTIC PROBLEMS

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THE LINEAR PROBLEM

We consider the following variational problem: Find $u \in W$ such that

$$a(u,v) = L(v) \equiv L^{\Omega}(v) + L^{\Gamma}(v) \qquad \forall v \in V \tag{1}$$

where

$$a(v,w) = \iint_{\Omega} k_{ij}\, v_{,i}\, w_{,j}\, dx, \tag{2}$$

$$L^{\Omega}(v) = \iint_{\Omega} vf\, dx, \qquad L^{\Gamma}(v) = \int_{\Gamma_2} vq\, ds, \tag{3}$$

$$V = \{v \in H^1(\Omega):\ v = 0 \text{ on } \Gamma_1;\ \text{mes}_1 \Gamma_1 > 0\},\ W = z + V; \tag{4}$$

Ω is a bounded domain in E_2 with a boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ ($\Gamma_1 \cap \Gamma_2 = \emptyset$); $z \in H^1(\Omega)$, $z = \bar{u}$ on Γ_1, $\bar{u} \in L_2(\Gamma_1)$ is a given function. In (2) and in what follows the summation convention over repeated subscripts is adopted and $v_{,i} = \partial v/\partial x_i$. The functions $k_{ij} = k_{ij}(x)$ are bounded and measurable in $\tilde{\Omega} \supset \bar{\Omega}$ and satisfy

$$k_{ij}(x)\xi_i\xi_j \geq C\xi_i\xi_i \qquad \forall x \in \tilde{\Omega} \supset \bar{\Omega} \qquad \forall \xi_i, \xi_j \in E_1, \tag{5}$$

where $C > 0$, and $f \in L_2(\Omega)$, $q \in L_2(\Gamma_2)$. Assumption (5) and Friedrichs' inequality imply that the form $a(v,w)$ is V-elliptic. Thus, according to the Lax-Milgram lemma, problem (1) has just one solution.

Problem (1) is approximated by the problem: Find $u_h \in W_h$ such that

$$a_h(u_h,v) = L_h(v) \equiv L_h^{\Omega}(v) + L_h^{\Gamma}(v) \qquad \forall v \in V_h \tag{6}$$

where V_h is a finite element approximation of V and $W_h = z_h + V_h$, z_h being a finite element approximation of z. The forms $a_h(v,w)$, $L_h(v)$ approximate the forms $a(v,w)$, $L(v)$ in the following way: The sets Ω and Γ_2 appearing in (2), (3) are substituted by Ω_h and Γ_{h2} and the obtained forms $\tilde{a}_h(v,w)$, $\tilde{L}_h(v)$ are then computed numerically.

In the Ciarlet's and Raviart's theory and its modifications (see [1],[2],[6]) the solution u of (1) is assumed to be sufficiently smooth, $u \in H^{n+1}(\Omega)$ $(n \geq 1)$, and the maximum rate of convergence is proved:

$$\|\tilde{u} - u_h\|_{1,\Omega_h} \leq Ch^n, \tag{7}$$

where $\tilde{u}$ is the Calderon's extension of u into $H^{n+1}(E_2)$.

The problem of convergence of u_h to $\tilde{u}$ (when $u \in H^1(\Omega)$ only) was solved recently in [8]. In this section we present outlines of considerations of [8].

Theorem 1. Let the boundary Γ of the domain Ω be piecewise of class C^{n+1}. Let Γ_h approximate Γ piecewise by arcs of degree n. Let $k_{ij}, f \in W_\infty^{(n)}(\tilde{\Omega})$ and let the quadrature formula used on the standard triangle T_0 for calculation of $a_h(v,w)$ and $L_h^{\Omega}(v)$ be of degree of precision $2n - 2$. Let $q \in C^n(\bar{U})$, where U is a domain containing Γ_2, and let the quadrature formula used on $[0,1]$ for calculation of $L_h^{\Gamma}(v)$ be of degree of precision $2n - 1$. Let $\bar{u}$ be so smooth that there exists a function $z \in H^2(\Omega)$ such that $z = \bar{u}$ on Γ_1. Then

$$\|\tilde{u} - u_h\|_{1,\Omega_h} \to 0 \quad \text{if} \quad h \to 0. \tag{8}$$

Proof. The assumptions concerning Γ and $a_h(v,w)$ imply, according to [2] and [7], that the forms $a_h(v,w)$ are uniformly V_h-elliptic. Thus we have similarly as in [1], [2]:

$$\|\tilde{u} - u_h\|_{1,\Omega_h} \leq C \left\{ \sup_{w \in V_h} \frac{|L_h(w) - \tilde{a}_h(\tilde{u},w)|}{\|w\|_{1,\Omega_h}} + \inf_{v \in W_h} \left[\|\tilde{u} - v\|_{1,\Omega_h} + \sup_{w \in V_h} \frac{|\tilde{a}_h(v,w) - a_h(v,w)|}{\|w\|_{1,\Omega_h}} \right] \right\}. \tag{9}$$

For simplicity we prove Theorem 1 only in the case $\Gamma_1 = \Gamma$, $\bar{u} = 0$ and $n = 1$, i.e. Ω_h has a polygonal boundary and we use linear triangular finite elements. As Γ is piecewise of class C^2 each boundary triangle satisfies (for sufficiently small h) one of the following possibilities:

$$\text{a)}\ \ T \subseteq T_{id}, \qquad \text{b)}\ \ T_{id} \subset T, \tag{10}$$

where T_{id} is the ideal boundary triangle (see [9]) whose approximation is T.

In order to estimate the first term on the right-hand side of (9) let us define a function $\hat{w} \in V$ associated with $w \in V_h$ in the following way: $\hat{w}(P_i) = w(P_i)$ at the vertices P_i of the triangles of the triangulation T_h of the domain Ω_h; the function $\hat{w}$ is continuous on $\bar{\Omega}$, linear on each interior triangle of T_h, equal to zero on $T_{id} - T$ and linear on T in the case (10a) and, finally, equal to Zlámal's ideal "linear" interpolate of w on T_{id} in the case (10b) (see [9]). Thus we can write, according to (1):

$$L_h(w) - \tilde{a}_h(\tilde{u},w) = L_h(w) - L(\hat{w}) + a(u,\hat{w}) - \tilde{a}_h(\tilde{u},w). \tag{11}$$

We have

$$L(\hat{w}) = \iint_{\Omega} f\hat{w}dx = \iint_{\Omega_h} fw\,dx + \sum_b \left\{ \iint_{T_{id}} (\hat{w} - w)f\,dx - \iint_{T-T_{id}} wf\,dx \right\}$$

where the sum is taken over all boundary triangles (10b). Expressing similarly $a(u,\hat{w}) - \tilde{a}_h(\tilde{u},w)$ we obtain from (11):

$$|L_h(w) - \tilde{a}_h(\tilde{u},w)| \le |\tilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)| +$$

$$+ \sum_b \left| \iint_{T_{id}} \left\{ (w - \hat{w})f + k_{ij}u,_i(\hat{w} - w),_j \right\} dx \right| +$$

$$+ \sum_b \left| \iint_{T-T_{id}} (wf - k_{ij}\tilde{u},_i w,_j)dx \right|. \tag{12}$$

According to [2, Theorem 4.1.5],

$$|\tilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)| \le Ch\|w\|_{1,\Omega_h}. \tag{13}$$

Denoting the first sum in (12) briefly by S_1 we have, according to the Cauchy inequality and the boundedness of k_{ij}:

$$|S_1| \le (\|f\|_{0,\Omega} + C\|u\|_{1,\Omega}) \left\{ \sum_b \|\hat{w} - w\|^2_{1,T_{id}} \right\}^{1/2}.$$

The definition of $\hat{w}$ and the proof of [9, Theorem 2] imply

$$\|w - \hat{w}\|_{1,T_{id}} \le Ch\|w\|_{2,T_{id}} \le Ch\|w\|_{1,T}$$

because w is linear on T and (10b) holds. Thus

$$|S_1| \le Ch\|w\|_{1,\Omega_h}. \tag{14}$$

As $\text{mes}(T - T_{id}) = O(h^3)$ we have

$$\sum_b \|w\|^2_{0,T-T_{id}} \le Ch^3 \sum_b \max w^2.$$

Further, similarly as [7,(39)] we can prove

$$\|w\|^2_{0,\Omega_h} \ge Ch^2 \sum_{T\in T_h} \sum_{i=1}^{3} [w(P_T^i)]^2$$

where P_T^i (i = 1,2,3) are the vertices of T. Using these results and the fact that $w_{,i}$ are constants on each $T \in \mathcal{T}_h$ we can easily find for the second sum S_2 on the right-hand side of (12):

$$|S_2| \cdot \|w\|_{1,\Omega_h}^{-1} \le C\left\{ \sum_b \|w\|_{0,T-T_{id}}^2 \|w\|_{0,\Omega_h}^{-2} + \right.$$

$$\left. + \sum_b |w|_{1,T-T_{id}}^2 |w|_{1,\Omega_h}^{-2} \right\}^{1/2} \le Ch^{1/2} . \qquad (15)$$

According to (12) - (15), the first term on the right-hand side of (9) is $O(h^{1/2})$.

As to the second term on the right-hand side of (9) we can find a set $\{v_h\}$, where $v_h \in W_h$, such that

$$\|\tilde{u} - v_h\|_{1,\Omega_h} \to 0 \quad \text{if } h \to 0. \qquad (16)$$

The following proof of (16) holds also in the case $\text{mes}_1 \Gamma_1 < \text{mes}_1 \Gamma$: The set $G = C^\infty(\bar{\Omega}) \cap V$ is dense in V (see [3]). Thus for every $\varepsilon > 0$ we can find $v_\varepsilon \in G$ such that $\|u - v_\varepsilon\|_{1,\Omega} < \varepsilon$. Let $\tilde{v}_\varepsilon$ and $\tilde{v}_\varepsilon^*$ be the Calderon's extensions of v_ε into $H^1(E_2)$ and $H^2(E_2)$, respectively. We have

$$\|\tilde{u} - I_h v_\varepsilon\|_{1,\Omega_h} \le \|\tilde{u} - \tilde{v}_\varepsilon\|_{1,\Omega_h} +$$

$$+ \|\tilde{v}_\varepsilon - \tilde{v}_\varepsilon^*\|_{1,\Omega_h-\Omega} + \|\tilde{v}_\varepsilon^* - I_h v_\varepsilon\|_{1,\Omega_h} \qquad (17)$$

where $I_h v_\varepsilon$ is the interpolate of v_ε in W_h, i.e. the piecewise linear function which has the same function values at the vertices of $T \in \mathcal{T}_h$ as the function v_ε. The properties of the Calderon's extensions, the absolute continuity of the Lebesgue integral and the finite element interpolation theorem imply that for all $h \le h_0(\varepsilon)$ the right-hand side of (17) is bounded by $K\varepsilon$, where K does not depend on ε. As $I_h v_\varepsilon \in V_h = W_h$ relation (17) implies (16).

The set $\{v_h\}$ appearing in (16) is bounded. Thus, according to [2, Theorem 4.1.4], the third term on the right-hand side of (9) is O(h). This finishes the proof in our simple case. The general case $\Gamma_1 \subset \Gamma$, $n \ge 1$ is considered in [8].

SOME NONLINEAR PROBLEMS

Let the form a(u,v) appearing in (1) be now nonlinear in u, linear in v, strongly monotone and Lipschitz continuous and let $a(0,v) = 0$ for all $v \in H^1(\Omega)$. In addition, let the forms $a_h(v,w)$ be uniformly strongly monotone and uniformly Lipschitz continuous in X_h (the finite element approximations of $H^1(\Omega)$), i.e. let

$$a_h(v,v-w) - a_h(w,v-w) \ge C|v-w|_{1,\Omega_h}^2 ,$$

$$|a_h(v,z) - a_h(w,z)| \le K\|v - w\|_{1,\Omega_h}\|z\|_{1,\Omega_h}$$

$$\forall v,w,z \in X_h \subset H^1(\Omega_h) \quad \forall h \in (0,h_0)$$

where the positive constants C, K do not depend on v,w,z and h. Finally, let the forms $\tilde{a}_h(v,w)$ be uniformly Lipschitz continuous. Under these assumptions the abstract error estimate has again the form (9) (see [5]).

A typical form $a(u,v)$ satisfying all assumptions presented in this section is given by relation (2) with

$$k_{ij} = b(x,(\nabla v)^2)\delta_{ij} \tag{18}$$

where δ_{ij} is the Kronecker delta and where the function $b(x,\eta)$ has the following properties (see [4]):

a) The functions $b(x,\eta)$, $\partial b(x,\eta)/\partial x_i$, $\partial b(x,\eta)/\partial\eta$ are continuous in $\tilde{\Omega} \times [0,\infty)$, where $\tilde{\Omega} \supset \bar{\Omega}$.

b) There exist constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \le b \le c_2, \quad |\partial b/\partial x_i| \le c_2, \quad 0 \le \partial b/\partial\eta \le c_2 \quad \text{in } \tilde{\Omega} \times [0,\infty),$$

$$|\xi|(\partial b/\partial\eta)(x,\xi^2), \quad \xi^2(\partial b/\partial\eta)(x,\xi^2) \le c_2 \quad \forall x \in \tilde{\Omega}, \quad \forall\xi \in E_1.$$

The functions (18) with properties a), b) appear in many physical and technical applications.

Now we generalize the result introduced in Theorem 1:

Theorem 2. Let the form $a(u,v)$ appearing in (1) be defined by (2) and (18) and let the function $b(x,\eta)$ have properties a), b). Let the assumptions of Theorem 1 be satisfied with $n = 1$. Then the solutions u and u_h of problems (1) and (6) exist and are unique and relation (8) holds. In addition, if $u \in H^2(\Omega)$ then the rate of convergence is given by (7), where $n = 1$.

Proof. As $n = 1$ we consider only linear triangular elements. We again restrict ourselves to the case $\Gamma_1 = \Gamma$, $\bar{u} = 0$. The existence and uniqueness of u and u_h is proved in [4]. The first property b) allows us to repeat (11) - (15). Thus the first term on the right-hand side of (9) is $O(h^{1/2})$. Also relation (16) remains unchanged, only the analysis of the third term on the right-hand side of (9) is different: As $\nabla v = \text{const.}$ on $T \in \mathcal{T}_h$ for all $v \in V_h = W_h$ we can write

$$|a_h(v_h,w) - \tilde{a}_h(v_h,w)| \le \sum_{T\in\mathcal{T}_h} |\text{mes}(T)b(P_T,g_h|_T) -$$

$$- \iint_T b(x,g_h)dx| \cdot |(\nabla v_h|_T \cdot \nabla w|_T)|$$

where v_h are the functions from (16) and $q_h = (\nabla v_h)^2$. We used one-point integration formula with the centre of gravity P_T of $T \in \mathcal{T}_h$. Using the properties of the function $b(x,\eta)$ we see, according to [2, Theorem 4.1.5], that the absolute value of the difference on the right-hand side is bounded by Ch mes(T). Thus the right-hand side of the last inequality is bounded by $Ch\|w\|_{1,\Omega_h}$ and relation (8) is valid.

The error estimate in the case $u \in H^2(\Omega)$ is derived in [5] where also more general forms a(v,w) are considered.

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