EQUADIFF 6

Rudolf Kodnár

A posteriori estimations of approximate solutions for some types of boundary value problems

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [373]--378.

Persistent URL: http://dml.cz/dmlcz/700159

Terms of use:

© Masaryk University, 1986

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

A POSTERIORI ESTIMATIONS OF APPROXIMATE SOLUTIONS FOR SOME TYPES OF BOUNDARY VALUE PROBLEMS

R. KODNÁR
Institute of Applied Mathematics, Comenius University
Mlynská dolina, 842 15 Bratislava, Czechoslovakia

1. Motive

When any approximate method is employed, it is of importance to know an estimation of the error involved in the approximate solution. In general, there exist two kinds of such estimations (i) a priori and (ii) a posteriori. The a priori assessments are obtained from the qualitative properties of the problem. They ussually possess an asymptotic character, are pessimistic and are used chiefly in theoretical considerations. The a posteriori assessments are carried out on the basis of already constructed approximate solution. Among the ways of construction of a posteriori estimations, a significant role is played by s.c. counter-direction methods. They are based on the following simple considerations:

Let A be a linear positively definite operator in a real Hilbert space H. Then the problem to find a generalized solution of the equation

$$Au=f, \quad f\in H \tag{1}$$

and minimization of the functional

$$F(u)=[u,u]_A-2(u,f)_H \tag{2}$$

are equivalent [1]. In (2) $[\,,]_A$ denotes a scalar product in the space H_A of the generalized solution to equation (1). It holds

$$\min_{H_A} F(u)=-\|u_0\|_A^2 ,$$

$$\|u_n-u_0\|_A^2=F(u_n)+\|u_0\|_A^2 ,$$

where u_n is an approximate solution constructed by the variational method and $\|\ \|_A$ is the norm in H_A. Usually the numbers d can be constructed greater than $\|u_0\|_A^2$ but close to it (lower bound estimation of $F(u_0)$). Then with their aid we get

$$\|u_n-u_0\|_A \leq (F(u_n)+d)^{1/2}. \tag{3}$$

Such a construction of numbers d is possible, for example, in the following case:

Let us assume that there exists a functional $F_1(u)$ such that

$$\inf_{H_A} F_1(u)=\|u_0\|_A^2 .$$

Then it is possible to select $d=F_1(u)$, $u\in H_A$. With the selection of u being suitable, it can be achieved that d is very near to $\|u_0\|_A^2$.

Inequality (3) explicitly gives the error estimation of the approximate solution. In the case of equations with worse operators than those mentioned in (1), it is not always possible to obtain an estimation in the form of (3), but one gets estimations of some other types. In literature several constructions of the lower estimations d are described. But the majority of them is applicable for the single special problems only.

Functions in the whole text are real.

2. Basic notions

Let Q be a bounded domain in R^m with the boundary ∂Q which satisfies several conditions of smoothness. Let on $\overline{Q}$ be given a boundary value problem such that $u_0\in V$ is its weak solution if

$$\forall v\in V: \ ((v,u_0))=\langle v,f\rangle+Z(v,h)-((v,w)). \tag{4}$$

Nearby $((,))$ is a bounded bilinear form on $H^k(Q)$, $k\geq 1$, $V\subset H^k(Q)$, $\langle \, , \rangle$ is a duality on V, $w\in H^k(Q)$,

$$Z(v,h)=\sum_{p=1}\sum_{l=1}\int_{Q_p}\frac{\partial^{t_{pl}} v}{\partial n^{t_{pl}}}h_{pl}dS,$$

$$\bigcup_1^r \partial Q_p = Q,$$

$f\in V^*$, $h_{pl}\in L_2(\partial Q_p)$, n is a direction an outward normal to Q. Let

$$\forall v\in V: \ ((v,v))\geq a^2\|v\|_V^2 \ , \ a \ R^1, \tag{5}$$

$$\forall u,v\in V: \ ((u,v))=((u,v)). \tag{6}$$

Theorem 1. Let

$$\forall u\in V, \forall v\in H^k(Q): \ ((u-v,u-v))\geq 0. \tag{7}$$

Let $v_1\in H^k(Q)$ be such that

$$\forall u\in V: \ ((u,v_1))=\langle u,f\rangle+Z(u,h)-((u,w)). \tag{8}$$

Then

$$\forall v_n \in V:\ a^2 \|u_0 - v_n\|_V^2 = ((v_n, v_n)) + 2\langle v_n, f\rangle - 2Z(v_n, h) + \\ + 2((v_n, w)) + ((v_1, v_1)), \tag{9}$$

holds.

Proof. Immediately follows from (5),(6),(7),(8).

If the informations on regularity of solution of the starting boundary value problem are avalaible construction of the lower estimation may be simplified. Consider a linear boundary value problem

$$Au = f \quad \text{in } Q, \tag{10}$$
$$B_i u = 0 \quad \text{on } \partial Q,$$

where A is a differential operator of 2k-th order. Denote

$$K = \{u \in C^{(2k-1)}(\overline{Q}) \cap C^{(2k)}(Q),\ Au \in L_2(Q)\},$$

$$D_A = \{u \,|\, u \in K,\ u \text{ fulfills all the boundary conditions from } (10)\}.$$

Suppose that A, f, Q and the boundary conditions in (10) are such, that the solution u_0 of the problem (10) belongs to D_A. Let $((,))_1$ be a symmetrical bilinear form such that

$$\forall u \in K:\ ((u,u))_1 \geqq 0,\ ((u,u))_1 = 0 \Leftrightarrow u = 0, \tag{11}$$
$$\forall u \in K, \forall v \in D_A:\ ((u,v))_1 = (Au, v)_{L_2}. \tag{12}$$

Remark 1. It can be shown that such a form exists for the most of the boundary value problem with Laplace and also with biharmonic operator. Form $((,))_1$ to the given problem is not uniquely defined.

Theorem 2. u_0 minimizes in D_A the functional

$$F_1(u) = ((u,u))_1 - 2(u,f)_{L_2}.$$

Let $v \in K$ be such that $Av = f$. Then

$$F_1(u_0) \geqq -((v,v))_1.$$

Proof. $\forall u \in D_A, \forall v \in K:\ ((v-u, v-u))_1 \geqq 0$. Then

$$((u,u))_1 - 2(u,f)_{L_2} \geqq -((v,v))_1 - 2(u,f)_{L_2} + 2((u,v))_1.$$

From it follows

$$\forall u \in D_A, \forall v \in K:\ F_1(u) \geqq -((v,v))_1 - 2(Av - f, u)_{L_2}.$$

Denote $J(v) = ((v,v))$. $J(v)$ is a functional defined on $H^k(Q)$. Let the assumptions of Theorem 1 be fulfilled. Then for $v \in H^k(Q)$ fulfilling (8) there is $J(v) \geqq ((u_0, u_0))$. Let

$$D_J = \{v \,|\, v \in H^k(Q) \text{ and fulfil } (8)\}.$$

Then $((u_0, u_0)) = \min\limits_{D_J} J(v)$.

Lemma 1. The minimizing sequence for the functional $J(v)$ converges to the solution u_0 of the equation (4) in the following sense

$((u_n-u_0,u_n-u_0))\longrightarrow 0.$

Proof. Denote the minimizing sequence $\{u_n\}_1^\infty$. Then $D_J\ni z_n=u_n-u_0$. It holds $\forall u\in V$: $((z_n,u))=0$. From that

$$((u_n-u_0,u_n-u_0))=((u_n,u_n))-2((z_n,u_0))-((u_0,u_0))=$$
$$=((u_n,u_n))-((u_0,u_0)).$$

Remark 2. Procedure formulated by Theorem 1,2 is a generalization of Trefftz method.

3. The construction of minimizing sequence

Denote

$$U=\{v\in H^k(Q)\,|\,\forall u\in V:\ ((u,v))=\langle u,f\rangle+Z(u,h)-((u,w))\}.$$

Lemma 2. The set U is convex and closed.

Proof. By direct verifying.

Lemma 3. The functional J(v) is convex on $H^k(Q)$ and its minimum on U is attained at u_0.

Proof. Convexity follows from differentiability.

Corollary 1. Relations

$$\forall v\in U:\ J(u)\leq J(v),$$
$$\forall v\in U:\ J'(u,v-u)\geq 0$$

are equivalent. The given problem can be solved by means of variational inequalities. The obtained minimizing sequence converges by given way to the solution u_0.

Thus we get further counter-direction methods to variational method of the solution of the primary problem.

4. Several special cases

Let A be a linear, positively definite operator $A\in(H,H)$. Let be given further Hilbert space H_1 with scalar product $(,)_1$. Let

$$A=T^*BT, \tag{13}$$

where T is operator $T\in(D_T\subset H,H_1)$, B is positively definite operator $B\in(H_1,H_1)$. $T^*\in(D_{T^*}\subset H_1,H)$ is operator adjoint to T. Assume that

$$D_T\supset D_A,\quad D_{T^*}\supset BTD_A.$$

Operator T^*BT is thus defined at least on D_A. Let the problem Au=f have solution $u_0\in D_A$, while $f=T^*g$, where $g\in D_{T^*}$. Denote

$$w_0=Tu_0,\quad w=Tu,\quad u\in D_T,$$

$$G(w)=(Bw,w)_1-2(g,w)_1 \ , \ w\in H_1.$$

Theorem 3. w_0 minimizes G on $TD_T\subset H_1$. If $T^*v=f$, then

$$G(w_0)= -\frac{1}{a}(v,v)_1, \tag{14}$$

where a is a constant from positively definiteness of operator B.

Remark 3. In case $A=\Delta\Delta$ and Dirichlet boundary conditions we get from Theorem 3 the principle of the method of unharmonic residue [1].

Corollary 2. $U_1=\{v\in D_{T^*}\,|\,T^*v=f\}$ is convex and closed set. $(v,v)_1$ is a convex functional on H_1. If there is $w_0\in U_1$ and $a\geq 1$ (from (14)), then w_0 minimizes functional $(v,v)_1$ on U_1.

Example. Consider a boundary value problem

$$\Delta\Delta u=f, \quad u\Big|_{\partial Q}=\frac{\partial u}{\partial n}\Big|_{\partial Q}=0, \ f\in L_2(Q). \tag{15}$$

Let the problem (15) have the solution $u_0\in C^{(1)}(\overline{Q})\cap C^{(4)}(Q)$. Denote:

$H=L_2(Q)=H_1$,

$D_A=\{u\,|\,u\in C^{(4)}(Q)\cap C^{(1)}(\overline{Q}),\ \Delta\Delta u\in H,$ u fulfil the boundary conditions from (15)$\}$,

$D_T=\{u\,|\,u\in C^{(1)}(\overline{Q})\cap C^{(4)}(Q),$ u fulfil the boundary conditions from (15)$\}$,

$D_{T^*}=\{v\,|\,v\in C^{(1)}(\overline{Q})\cap C^{(2)}(Q),\ \Delta v\in H\}$.

Define

$$\forall u,v\in D_{T^*}:\ (u,v)_1=\sum_{|i|\leq 2}\int_Q D^iuD^iv\ dQ=(u,v)_{H^2}\ .$$

Then there is in (13) $T=T^*=\Delta$, B is an identic operator. On lower estimations of minimum of the functional

$$u\in D_A:\ (\Delta\Delta u,u)_{L_2}-2(u,f)_{L_2}$$

Theorem 3 and Corollary 2 can be used.

Remark 4. Problem (15) is a mathematical model of clamped plate. Similarly the mathematical models of further kinds of boundary of plate and web we may investigated [2].

Remark 5. Procedures from sections 2,3,4 may also be used for non-linear problem of the special type:

$u_0\in V$ is called the solution of the problem if

$f(x,u_0+w)\in L_2(Q)$,

$$\forall v\in V:\ ((v,u_0))+\int_Q f(x,u_0(x)+w(x))v(x)\ dQ=Z(v,h)-((v,w)).$$

At the same time it is supposed that function $f:\ \overline{Q}\times R^1\rightarrow R^1$ is continuous and for fixed $x\in Q$

$$f(x,r_1)\leq f(x,r_2)\ \ \forall r_1,r_2\in R^1,\ \ r_1\leq r_2$$

holds.

All the rest notations are the same as in (4).

5. Slobodyanskii procedure

In [3] Slobodyanskii proposed procedure to get lower bound assessment. Generalization of this procedure for further, even nonlinear problems, is described in [4].

REFERENCES

[1] Michlin, S.G.: Variationsmethoden der Mathematischen Physik. Berlin: Akademie Verlag 1962.

[2] Kodnár, R.: A posteriori estimation of approximate solution for mathematical models of plate and web. (in preparation).

[3] Slobodyanskii M.G.: On corvension of the problem of minimum of a functional into the problem of maximum. Doklady Akad. Nauk SSSR 91, Nr.4 (1953) (Russian).

[4] Djubek, J., R.Kodnár, M.Škaloud: Limit state of the plate elements of steel structures. Basel-Boston-Stuttgart: Birkhäuser Verlag 1984, 298p.