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Jiří Neustupa

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THE GLOBAL EXISTENCE OF WEAK SOLUTIONS OF THE MOLLIFIED SYSTEM OF EQUATIONS OF MOTION OF VISCOUS COMPRESSIBLE FLUID

J. NEUSTUPA
Faculty of Mechanical Engineering, Czech Technical University
Suchbátarova 4, 166 07 Prague 6, Czechoslovakia

1. Introduction

It is known that weak solutions of the Navier-Stokes equations for incompressible liquid exist on a time interval of an arbitrary lenght (see e.g. [3] [10]). No analogous result has been derived in the case of equations of motion of viscous compressible fluid till now. Only the existence of solutions of such equations local in time was proved (see e.g. [1], [5], [8], [9]) and if some theorems about the global in time existence of solutions appeared, they contained assumptions of the type "the initial conditions are small enough" (see e.g. [4]), "the flow is one-dimensional" ([2]), etc. We study the existence of weak solutions of the equations of motion of viscous compressible fluid on a time interval of a given lenght in this paper, but the system of equations we deal with is rather modified in a comparison with a full general system of equations governing the motion of viscous compressible fluid. The modification consists in the following points:

a) We assume the dynamic viscosity coefficient μ to be a positive constant.

b) We do not take the energy equation into account and we use the relation between the pressure p and the density

$$p = c.\widetilde{\rho^{\varkappa}} \tag{1.1}$$

instead of it. c and $\varkappa$ are constants such that $c > 0$, $\varkappa \in (1,6)$. The tilda over $\rho^{\varkappa}$ represents a certain regularization (mollification). Its exact meaning is explained in the paragraph 2., but we can write in advance that $\widetilde{\rho^{\varkappa}}(x)$ is an average of $\rho^{\varkappa}$ considered with a proper smooth weight function on a neighbourhood $B_h(x)$ of x (where the radius h of this neighbourhood may be arbitrarily small).

c) We use the mollification denoted by ~ also in some terms in the Navier-Stokes equations for the system we deal with has the form

(1.2) $\rho_{,t} + (\rho\tilde{u}_j)_{,j} = 0$,

(1.3) $(\rho u_i)_{,t} + (\rho\tilde{u}_j u_i)_{,j} = - c.(\widetilde{\rho^{\varkappa}})_{,i} + \frac{1}{3}\mu u_{j,ji} + \mu u_{i,jj}$

$(i = 1,2,3)$.

$U = (u_1, u_2, u_3)$ has a physical meaning of the velocity of the moving fluid. In [7], R. Rautmann used the similar mollification in the Navier-Stokes equations for the incompressible liquid in order to prove the global in time existence of strong solutions in three-dimensional domains. The notion of the velocity of the fluid at the point x is usually introduced by means of an average of the velocities of all particles of the fluid contained in a small neighbourhood of x. So if h is small enough, $\tilde{u}_i$ is almost the same as u_i from the point of view of mechanics. The system (1.3) expresses the 2nd Newton law of mechanics applied to particles moving along the integral curves of the flow field $\tilde{U}$.

We shall use the Rothe method. We can give only a brief outline of the whole procedure here. Details may be found in [6].

2. Formulation of an initial-boundary value problem

Assume that Ω is a bounded region in R^3 with the boundary of the class $C^{2+(\alpha)}$ for some $\alpha \in (0,1)$. Let us choose $h > 0$ and put

$$\Omega_h = \{x \in R^3;\ \mathrm{dist}(x,\Omega) < h\} .$$

Assume that h can be chosen so small that $\partial\Omega_h$ is also of the class $C^{2+(\alpha)}$. Put

$$\omega_h(\xi) = K_h \exp\left(-\frac{|\xi|^2}{h^2-|\xi|^2}\right) \text{ for } \xi \in R^3,\ |\xi| < h ,$$

$$\omega_h(\xi) = 0 \quad \text{for } \xi \in R^3,\ |\xi| \geq h .$$

Let K_h be chosen so that the integral of ω_h over R^3 is equal to 1. If $f \in L^1(\Omega_h)$, put

(2.1) $$\tilde{f}(x) = \int_{\Omega_h} \omega_h(x - y)f(y)dy .$$

If f is defined in $\Omega_h \times R^1$ then we denote by $\dot{f}$ the function regularized in the space variable only. If the regularization ~ is applied to any function def in the space variable on Ω only (like for example components of the velocity or their approximations), we deal with this function as if it is defined on Ω_h and is identically equal to zero on $\Omega_h - \Omega$.

We shall solve the equation (1.2) on $\Omega_h \times (0,T)$ and the system (1.3) on $\Omega \times (0,T)$ (where T is a given positive number). We consider the boundary condition

(2.2) $$u_i\big|_{\partial\Omega} \equiv 0 \quad (i = 1,2,3)$$

and the initial conditions

(2.3) $$\rho\big|_{t=0} = \rho_0 ,$$

(2.4) $$(\rho u_i)\big|_{t=0} = \rho_0 u_{0i} \quad (i = 1,2,3) ,$$

where ρ_0, $U_0 = (u_{01},u_{02},u_{03})$ are given functions such that $\rho_0 \in H^1(\Omega_h)$, $\rho_0 \geq 0$, $U_0 \in \mathring{H}^1(\Omega)^3$.

We shall call by the weak solution of (1.2), (1.3), (2.2), (2.3), (2.4) the couple of functions U,ρ such that

(2.5) $$U \equiv (u_1,u_2,u_3) \in L^2(0,T;\mathring{H}^1(\Omega)^3), \quad \rho \in L^\infty(0,T;H^1(\Omega_h)), \ \rho \geq 0 ,$$

(2.6) $$\int_0^T \int_\Omega \{\rho u_i \varphi_{i,t} + \rho\tilde{u}_j u_i \varphi_{i,j} + c(\tilde{\rho}^\varkappa)\varphi_{i,i} - \tfrac{1}{3}\mu u_{j,j}\varphi_{i,i} - \mu u_{i,j}\varphi_{i,j}\}\,dxdt = - \int_\Omega \rho_0 u_{0i}(\varphi_i\big|_{t=0})\,dx$$

for all $\varphi \equiv (\varphi_1,\varphi_2,\varphi_3) \in C^\infty(\overline{\Omega} \times \langle 0,T\rangle)^3$ such that $\varphi_i\big|_{\partial\Omega} \equiv 0$, $\varphi_i\big|_{t=T} \equiv 0 \quad (i = 1,2,3)$,

(2.7) $$\int_0^T \int_{\Omega_h} \{\rho\psi_{,t} + \rho\tilde{u}_j\psi_{,j}\}\,dxdt = - \int_{\Omega_h} \rho_0(\psi\big|_{t=0})\,dx$$

for all $\psi \in C^\infty(\overline{\Omega}_h \times \langle 0,T\rangle)$ such that $\psi\big|_{t=T} \equiv 0$.

By means of a similar method as it is used in [1] in the case of the Navier-Stokes equations for the incompressible liquid, it can be proved that if U,ρ satisfy (2.5), (2.6), (2.7) then $\rho.U$ is a.e. in $\langle 0,T\rangle$ equal to a continuous function from $\langle 0,T\rangle$ into $H^{-1}(\Omega)^3$. Hence we can understand under $(\rho u_i)\big|_{t=0}$ (i=1,2,3) in (2.4) limits as $t \to 0+$ of the components of this function. Similarly, it may be shown that ρ is a.e. in $\langle 0,T\rangle$ equal to a continuous function from $\langle 0,T\rangle$ into $H^1(\Omega_h)^*$ (the dual of $H^1(\Omega_h)$). It gives a reasonable sence to the initial condition (2.3).

3. The time discretization

Let m be a natural number. Put $\tau = T/m$, $t_k = k.\tau$ $(k = -1,0,1,\ldots$ $\ldots,m)$. Denote $\rho^{(-1)} = \rho_0$, $u_i^{(0)} = u_{0i}$ $(i = 1,2,3)$ and let $\rho^{(k)}$, $U^{(k)} = (u_1^{(k)}, u_2^{(k)}, u_3^{(k)})$ denote an approximation of a solution on k-th time layer. A discrete version of (2.5), (2.6) and (2.7), which we use in the following, is: We look for $\rho^{(0)}, \rho^{(1)},\ldots,\rho^{(m)} \in H^1(\Omega_h)$, $\rho^{(k)} \geq 0$ $(k = 0,1,\ldots,m)$ and $U^{(1)},\ldots,U^{(m)} \in \mathring{H}^1(\Omega)^3$ so that

$$(3.1)_k \qquad \int_\Omega \{\rho^{(k-1)}u_i^{(k)}\Phi_i - \rho^{(k-2)}u_i^{(k-1)}\Phi_i - \tau\rho^{(k-1)}\tilde{u}_j^{(k-1)}u_i^{(k-1)}\Phi_{i,j}$$

$$- \tau c\tilde{\rho}^{(k)\varkappa}\Phi_{i,i} + \frac{1}{3}\tau\mu u_{j,j}^{(k)}\Phi_{i,i} + \tau\mu u_{i,j}^{(k)}\Phi_{i,j}\}dx = 0$$

for all $\Phi \equiv (\Phi_1,\Phi_2,\Phi_3) \in \mathcal{E}^\infty(\bar{\Omega})^3$ and $k = 1,\ldots,m$,

$$(3.2)_k \qquad \rho^{(k)} - \rho^{(k-1)} + \tau(\rho^{(k)}\tilde{u}_j^{(k)})_{,j} = 0$$

for $k = 0,1,\ldots,m$.

We can further proceed in such a way that we successively solve $(3.2)_0$ (for the unknown $\rho^{(0)}$), $(3.1)_1$ and $(3.2)_1$ (for the unknowns $U^{(1)},\rho^{(1)}$), ..., $(3.1)_m$ and $(3.2)_m$ (for the unknowns $U^{(m)}$, $\rho^{(m)}$). It can be done using standart methods of the functional analysis and the theory of the partial differential equations. The following inequalities may be also derived:

$$(3.3) \qquad \int_\Omega\{\frac{1}{2}\rho^{(k-1)}u_i^{(k)}u_i^{(k)} + \frac{1}{2}\sum_{s=1}^{k}\rho^{(s-2)}(u_i^{(s)}-u_i^{(s-1)})(u_i^{(s)}-u_i^{(s-1)}) +$$

$$+ \frac{1}{3}\tau\mu\sum_{s=1}^{k}(u_{j,j}^{(s)})^2 + \tau\mu\sum_{s=1}^{k}u_{i,j}^{(s)}u_{i,j}^{(s)}\}dx + \frac{c}{\varkappa-1}\int_{\Omega_h}\rho^{(k)\varkappa}dx \leq$$

$$\leq \int_\Omega\frac{1}{2}\rho_0u_{0i}u_{0i}dx + \frac{c}{\varkappa-1}\int_{\Omega_h}\rho_0^\varkappa dx \qquad (k = 1,\ldots,m),$$

$$(3.4) \qquad \|\rho^{(k)}\|^2_{H^1(\Omega_h)} + \sum_{s=0}^{k}\|\rho^{(s)} - \rho^{(s-1)}\|^2_{H^1(\Omega_h)} \leq$$

$$\leq K_1\exp(4\tau\|U_0\|_{L^2(\Omega)^3} + \int_\Omega\frac{1}{2}\rho_0u_{0i}u_{0i}dx +$$

$$+ \frac{c}{\varkappa-1}\int_{\Omega_h}\rho_0^\varkappa dx).\|\rho_0\|^2_{H^1(\Omega_h)} \qquad (k = 0,1,\ldots,m)$$

for an appropriate positive constant K_1, independent on k.

4. An approximate solution of (2.6), (2.7) and the limit process for $m \to +\infty$

Put

$$(4.1)\qquad {}^m\rho(t) = \rho^{(k)} \text{ for } t \in (t_k, t_{k+1}\rangle \quad (k = -1,0,1,\dots,m-1)\,,$$

$$(4.2)\qquad {}^mU(t) = U^{(k+1)} \text{ for } t \in (t_k, t_{k+1}\rangle \quad (k = 0,1,\dots,m-1)\,.$$

It follows from (3.3) and (3.4) that the sequence $\{{}^m\rho\}$ (resp. $\{{}^mU\}$) is uniformly bounded in $L^\infty(0,T;\ H^1(\Omega_h))$ (resp. in the space $L^2(0,T;\ \mathring{H}^1(\Omega)^3)$) and that $\{{}^m\rho|{}^mU|^2\}$ is uniformly bounded in $L^\infty(0,T;\ L^1(\Omega))$. Using the Hölder inequality, it can be also easily shown that $\{{}^m\rho\,{}^mU\}$ is uniformly bounded in $L^\infty(0,T;\ L^{12/7}(\Omega)^3)$ and in $L^2(0,T;\ W^1_{3/2}(\Omega)^3)$. There exist subsequences (denoted by $\{{}^m\rho\}$, $\{{}^mU\}$ again) and functions ρ, U so that ${}^m\rho \to \rho$ weakly - * in $L^\infty(0,T;\ H^1(\Omega_h))$, ${}^mU \to U$ weakly in $L^2(0,T;\ H^1(\Omega)^3)$, ${}^m\rho\ {}^mU \to \rho\ U$ weakly - * in $L^\infty(0,T;\ L^{12/7}(\Omega)^3)$ and weakly in the space $L^2(0,T;\ W^1_{3/2}(\Omega)^3)$. By means of other estimates of ${}^m\rho\ {}^mU$ and ${}^m\rho$ in $\mathcal{H}^\gamma(0,T;\ W^1_{3/2}(\Omega)^3,\ H^{-1}(\Omega)^3)$ and $\mathcal{H}^\gamma(0,T;\ H^1(\Omega_h),\ L^2(\Omega_h))$ (see e.g. [3] or [10] for the definition of these spaces), we can prove that even ${}^m\rho \to \rho$ strongly in $L^2(0,T;\ L^2(\Omega_h))$ and ${}^m\rho\ {}^mU \to \rho\ U$ strongly in $L^2(0,T;\ L^2(\Omega)^3)$.

The functions ${}^m\rho$, ${}^mU$ satisfy (2.6), resp. (2,7) with some errors E_1, resp. E_2. It is shown in [6] that $E_1 = O(\tau^{1/2})$ and $E_2 = O(\tau^{1/2})$ for $\tau \to 0+$ (i.e. $m \to +\infty$). These relations together with the types of convergences mentioned above are sufficient to prove that ρ, U satisfy (2.5), (2.6), (2.7).

If we use (3.3) and (3.4), we can also derive the estimate

$$(4.3)\qquad \|\rho\|^2_{L^\infty(0,T;\ H^1(\Omega_h))} \le K_1 \exp\Big(\int_\Omega \tfrac{1}{2}\rho_0 u_{0i} u_{0i}\,dx + \frac{c}{\varkappa-1}\int_{\Omega_h} \rho_0^\varkappa\,dx\Big).\|\rho_0\|^2_{H^1(\Omega_h)}$$

and the energy inequality

$$(4.4)\qquad \int_\Omega \tfrac{1}{2}\rho u_i u_i|_{t=t_1}dx + \frac{c}{\varkappa-1}\int_{\Omega_h}\rho^\varkappa|_{t=t_1}dx + \int_0^{t_1}\int_\Omega\{\tfrac{1}{3}\mu(u_{j,j})^2 + \mu u_{i,j}u_{i,j}\}\,dxdt \le$$
$$\le \int_\Omega \tfrac{1}{2}\rho_0 u_{0i}u_{0i}\,dx + \frac{c}{\varkappa-1}\int_{\Omega_h}\rho_0^\varkappa\,dx$$

(for every $t_1 \in \langle 0,T\rangle$).

While the estimate (4.3) depends on the parameter h (used in the reqularization in (1.2) and (1.3)) according to the dependance of K_1 on h, the energy inequality (4.4) is quite independent on h. But in spite of this fact, we are not able to prove that if $h \to 0+$, we can get a solution of (1.2), (1.3) without the mollification yet.

References

[1] ITAYA,N., *On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluids*, Kodai Math. Sem. Rep. 23, 1971, 60-120.

[2] KAZHIKOV,A.V., SHELUKIN,V.V., *Unique global solution in time of initial-boundary value problems for one-dimensional equations of a viscous gas*, Prikl. Math. Mech. 41, 1977, 282-291.

[3] LIONS,J.L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod. Paris. 1969.

[4] MATSUMURA.A.. NISHIDA.T.. *Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Comm on Math. Physics 89, 1983, 445-464.

[5] NASH.J.. *Le problème de Cauchy pour les équations diff. d'un fluide général*, Bull. Soc. Math. France 90, 1962, 487-497.

[6] NEUSTUPA,J.. *The global weak solvability of an initial-boundary value problem of the Navier-Stokes type for the compressible fluid*, to appear.

[7] RAUTMANN,R., *The uniqueness and regularity of the solutions of Navier-Stokes problems*, Funct. Theor. Meth. for PDR, Proc. conf. Darmstadt 1976, Lecture N. in Math., 561, 1976, 378-393.

[8] SOLONNIKOV,V.A., *Solvability of the initial boundary value problem for the equations of motion of a viscous compressible fluid*, J. Soviet Math. 14, 1980, 1120-1133.

[9] TANI,A., *On the first initial-boundary value problem of compressible viscous fluid motion*, Publ. RIMS Kyoto Univ. 13, 1977, 193-253.

[10] TEMAM,R., *Navier-Stokes equations*, North-Holland Publ. Comp., Amsterdam - New York - Oxford, 1979.