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RECENT RESULTS IN THE APPROXIMATION OF FREE BOUNDARIES

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§ 1. We present here a short survey on results recently obtained in the approximation of free boundaries. For examples of free boundary problems that are interesting in physics and engineering we refer for instance to |1|,|7|,|8|. Here we shall stay at a very abstract level, without considering, essentially, the nature of the free boundary problem under consideration nor the type of discretization which is employed. In that we are rather following |3| or the first part of |9|. For practical cases in which the following results apply we refer to |4|, |10| and |9|.

In the next section we present the framework in which the theory will be developed and in the third section we shall present some abstract results, most of them without proof. The proofs can be found in the corresponding references.

§ 2. For the sake of simplicity we shall consider the following "model" situation. We are given a bounded domain D in $\mathbb{R}^n$ with piecewise Lipschitz boundary (to fix the ideas). We are also given a function u(x) in $C^\circ(\bar{D})$. The function u(x) will be the solution of our free boundary problem. The nature of the problem itself is immaterial at this stage. We assume that

(1) $\quad u(x) \geq 0 \qquad \forall x \in \bar{D}$

and we assume that the continuous free boundary F is characterized by

(2) $\quad F := D \cap \partial(D^+)$

where

(3) $\quad D^+ := \{x \mid x \in \bar{D} \ , \ u(x) > 0\};$

We assume, finally that we have constructed a sequence $\{u_h(x)\}$, for $0 < h \leq h_0$ of "approximating solutions", which converges to u(x) in $C^\circ(\bar{D})$: Again, the procedure employed to construct $\{u_h\}$ is irrelevant at the moment. We set

(4) $\quad E_p(h) := \|u - u_h\|_{L^p(D)} \qquad 1 \leq p \leq \infty$.

and we remark that we have already assumed

(5) $\lim_{h\to 0} E_\infty(h) = 0$

We would like to construct a "discrete free boundary" F_h as in (2) and then to estimate the distance of F_h from F in terms of $E_p(h)$, defined in (4). In order to make our life even easier, we assume that, as in (1),

(6) $u_h(x)\geq 0 \quad \forall x\in\bar{D}, \quad \forall h\leq h_0$

and we set, as a first trial,

(7) $D_h^+:=\{x \mid x\in\bar{D},\ u_h(x)>0\}$

(8) $F_h:=\bar{D}\cap\partial(D_h^+)$

Unfortunatly, elementary examples show that F_h can be very far from F even for u_h very close to u. For instance if $D=|-1,1|$ and $u(x)=(x)^+$ (that is $u(x)=0$ for $x<0$ and $u(x)=x$ for $x>0$) we have $F=\{0\}$. If now $u_h(x)=u(x)+h^s(x+1)$ the $F_h=\{-1\}$ no matter how small is h or how big is s. It should be clear now that the setting (1)...(8) does not allow the proof of any bounds on the distance of F_h from F. In the next section we shall present a few remedies that (under suitable additional assumptions) have been proposed to improve the situation.

§ 3. The first trial in this direction has been done about ten years ago in |2|. Assume that g(h) is a function of h such that

(9) $E_\infty(h)<g(h) \quad 0<h\leq h_1\leq h_0$

(10) $\lim_{h\to 0} g(h)=0$

and set

(11) $D_{h,g}^+=\{x \mid x\in\bar{D},\ u_h(x)>g(h)\}$.

Then we have |2|.

<u>Theorem 1</u>-<u>Under the above assumptions we have, for</u> $h<h_1$

(12) $D_{h,g}^+\subset D^+$,

<u>and</u>, <u>for all</u> $x\in D^+$, <u>there exists</u> $h_2>0$ <u>such that</u> $x\in D_{h,g}^+$ <u>for</u> $h<h_2$.

The proof is immediate. We point out that, in other words, $D_{h,g}^+$ converges to D^+ "from inside". At our knowledge theorem 1 is still among the best result that one can obtain without additional information.

A tipical additional information that one can get, in many cases, is the behaviour of u(x) in D^+, near the free boundary. To fix the ideas, assume from now on that F is a smooth (say, C^1) surface. For any point $\bar{x}$ of F one can look at the restriction of u along the direction normal to F and pointing into D^+. If

(13) $u(x) \geq c(x-\bar{x})^s$ for $|x-\bar{x}| \leq a$ $(s \in \mathbb{R})$

with c,a and s independent of $\bar{x}$ we shall say that "u(x) grows like d^s near F" (d is for distance to the free boundary). For instance in the particular case of a nice obstacle problem, it has been proved in |5| that u(x) grows like d^2 near F. If one can prove a similar property for the approximations $u_h(x)$, one can use such an information to get estimates on the distance of F_h from F. This has been done in |4| for the case of a nice obstacle problem with a piecewise linear finite element approximation that satisfies the discrete maximum principle. The result in |4| is essentially that the distance of F_h from F behaves like $(E_\infty(h))^{\frac{1}{2}}$. This idea has then been extended in |10| to the one-phase Stefan problem in several dimensions, and in |11| to parabolic variational inequalities of obstacle type. The major drawback of this technique, however, is that it is often very difficult to prove growth properties for the discrete solutions $u_h(x)$, while the behaviour of u(x) itself is easier to analyze (see |8|,|7| for several growth properties proved on the continuousproblem).

A new and interesting set of results has been then obtained in |9| by combining, somehow, the two previous techniques: to change D_h^+ into a suitable $D_{h,g}^+$ and to use some growth property on u(x). In order to give the flavour of this procedure we shall present here two results in this direction. More detailed results and examples can be found in |9|.

Theorem 2 |9| With the notations and assumptions (1)...(11), if u(x) has the growth property (13) for some s>0, then there exist c>0 and $h_3>0$ such that for all h with $0<h\leq h_3$

(14) $\text{dist}\,(F_{h,g},F) \leq (cg(h))^{1/s}$,

where

(15) $F_{h,g} := D \cap \partial(D_{h,g}^+)$

and (14) means

(16) $\forall\, \bar{x}_h \in F_{h,g}\ \exists\, \bar{x} \in F$ such that $|\bar{x}-\bar{x}_h| \leq (cg(h))^{1/s}$.

Proof - Let $\bar{x}_h \in F_{h,g}$. This implies $u_h(\bar{x}_h)=g(h)>E_\infty(h)$, and therefore $\bar{x}_h \in D^+$. From (13) one has now that, for h small enough,

(17) $u(\bar{x}_h) \geq c|\bar{x}-\bar{x}_h|^s$

for some $\bar{x}\in F$. Since $u_h(\bar{x}_h)=g(h)$ one has from the triangle inequality

(18) $u(\bar{x}_h)\leq g(h)+E_\infty(h)<2g(h)$

and (16) follows from (17) and (18).

Theorem 3 Under the same assumptions of theorem 2, if

(19) $g(h)>E_p(h)^{sp/(1+sp)}$

then there exist $c>0$ and $h_4>0$ such that

(20) $\text{meas}(D^+\Delta D^+_{h,g})\leq c(g(h))^{1/s}$

where in (20) the symbol Δ indicates as usual the symmetric difference of sets.

The proof can be found in |9|.

We would like to conclude with a somehow phylosophical remark (see |3|). In general, condition (13) implies that the global regularity of u(x) in $\bar{D}$ is, at best

(21) $u\in C^s(\bar{D})$.

If one uses, for instance, finite element methods in order to approximate u, one has, at best (see e.g. |6|):

(22) $E_\infty(h)=\|u-u_h\|_{C^\circ}\leq ch^r\|u\|_{C^r}$ $0\leq r\leq \min(k+1,s)$

where k is the degree of the polynomials. Using now theorem 2, say, with $g(h)=2E_\infty(h)$ one has from (14)

(23) $\text{dist}\ (F_{h,g},F)\leq c(E_\infty(h))^{1/s}$

and from (23) and (22):

(24) $\text{dist}\ (F_{h,g},F)\leq ch^{r/s}$ (at best).

One can now make the following observations.

1) Since $r\leq s$ in (24), the error cannot beat the mesh size.

2) For s "big" one need a "big" k (that is, polynomials of high degreee) so that the case r=s can be achieved in (22) (and hence in (24)).

3) For s "small" the error that one gets from (24) is surprising good. Unfortunately, for irregular u, (22) is often difficult to prove in practical cases.

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