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ON NONLINEAR ELLIPTIC EQUATIONS IN INFINITE CYLINDERS

Angel Calsina — Joan Solà-Morales — Marta València

ABSTRACT. This communication presents a summary of the results of the paper [1]. Semilinear elliptic equations of second order in cylindrical domains are shown to define well posed initial value problems under some assumptions on the nonlinearity and for globally bounded solutions. Conditions to have a dynamical system and a compact global attractor are given as well as conditions for all the solutions approaching zero.

1. Introduction

Let Ω' be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let us define

$$\Omega := (s_0, \infty) \times \Omega' \subset \mathbb{R}^{n+1}$$

as our basic cylindrical domain, for some $s_0 \in \mathbb{R}$. We consider the problem of existence of classical solutions u (that is, $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$) of the following equation and additional conditions

$$\begin{aligned}
&\sum_{i,j=0}^{n} a_{ij}(x) u_{x_i x_j} + f(\nabla u, u, x) = 0 \qquad \text{for} \quad x := (x_0, x_1, \dots, x_n) \in \Omega\,, \\
&u(x) = 0 \qquad \text{for} \quad (x_1, x_2, \dots, x_n) \in \partial\Omega'\,, \\
&u(x) = \varphi(x_1, x_2, \dots, x_n) \qquad \text{for} \quad x_0 = s_0 \quad \text{and some} \quad \varphi \in \mathcal{C}_0^0(\overline{\Omega}')\,, \\
&|u(x)| \quad \text{globally bounded.}
\end{aligned} \tag{1}$$

Our aim is to have conditions on the nonlinearity $f(p, z, x)$ for this to be a well posed problem, in the sense of initial value problems, in the space $\varphi \in \mathcal{C}_0^0(\overline{\Omega}')$

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and in particular to have existence and uniqueness of solutions. Theorem 1 below gives results in this direction.

We also want to see if the approach that is typical in the theory of infinite-dimensional dynamical systems is applicable to this problem, and specifically the existence of a compact global attractor for the initial value problem considered above. These results are summarized in Theorem 2.

Finally, in Theorem 3 we give some sufficient conditions for all the solutions to tend to zero when x_0 tends to infinity.

Our work has to be seen as a partial attempt, with the methods of dynamical systems theory, to characterize the possible behaviour at infinity of the bounded solutions of some elliptic equations in cylindrical domains. In this sense, this spatially-asymptotic behaviour is somehow summarized in the global attractor of Theorem 2 below, a concept mainly developed in the recent years to deal with the time-asymptotic behaviour.

The use of a dynamical systems approach to this kind of problems is not new and has been used to obtain important results in these directions by several authors. We mention, for example, the papers [2, 3, 4], and [5].

In the paper [5] bounded solutions were obtained, by invariant manifold techniques, among all the solutions. These techniques impose restrictions to the Lipschitz constant of the nonlinearity. But in the present results these are merely one-sided (sign) conditions: roughly speaking they are upper bounds on $\partial f/\partial u$ instead of $|\partial f/\partial u|$. In comparison to [5] we use here classical techniques like the maximum principle instead of an abstract functional approach. This induces some limitations on the present results but we can now deal with more general (scalar) equations and nonlinearities, mostly regarding the nonlinear dependence of f on ∇u.

2. Existence of solutions

Existence and uniqueness of solutions to problem (1) is not an automatic property, but requires important limitations for $f(p,z,x)$ in order to achieve a delicate balance among convection, reaction and dissipation. Our hypotheses are the following:

H1: smoothness. The boundary $\partial\Omega'$ is of class $C^{2+\gamma}$ for some $\gamma \in (0,1)$. The functions $a_{ij}(x)$ are of class $C^{1+\gamma}$ in $\overline{\Omega}$, and the function $f(p,z,x)$ is of class C^1 in $\mathbb{R}^{n+1} \times \mathbb{R} \times \overline{\Omega}$.

H2: ellipticity. There exists a continuous function $\nu\colon [s_0,\infty) \to (0,\infty)$ such that the inequality

$$\sum_{i,j=0}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu(x_0)\sum_{i=0}^{n}\xi_i^2$$

holds for all $x \in \overline{\Omega}$ and $(\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$.

H3: balance. There exists an $\varepsilon > 0$ such that

$$\varepsilon^2 a_{00}(x) + \varepsilon \frac{\partial f}{\partial p_0}(p,z,x) + \frac{\partial f}{\partial z}(p,z,x) \leq 0 \tag{2}$$

for all $(p,z,x) \in \mathbb{R}^{n+1} \times \mathbb{R} \times \overline{\Omega}$.

H4: dissipation. There exists an $M_0 > 0$ such that

$$zf(0,z,x) < 0$$

for $|z| \geq M_0$ and $x \in \overline{\Omega}$.

H5: growth.

$$\big|f(p,z,x)\big| = O\big(|p|^2\big)$$

uniformly for $x \in \overline{\Omega}$ and bounded z.

We are now in the position to state the main result of this section:

THEOREM 1. *Let the hypotheses H1 to H5 be satisfied. Then the following conclusions hold:*

i) *For any initial condition $\varphi \in C_0^0(\overline{\Omega}')$ problem (1) has one and only one classical solution.*
ii) *The solution $u(x)$ depends on the initial condition φ in a continuous way, in the uniform norm for φ and in the topology of uniform convergence on compact sets of $\overline{\Omega}$ for $u(x)$.*
iii) *Solutions are strongly order-preserving in the following sense: Let $\varphi(y)$ and $\psi(y)$ be the initial conditions of $u(x)$ and $v(x)$ respectively. If $\varphi(y) \leq \psi(y)$ for all $y \in \Omega'$, then $u(x) < v(x)$ for all $x \in \Omega$, unless $u \equiv v$.*

Let us present now an example in order to discuss the meaning of hypothesis H3. Consider the equation $\Delta u + bu_{x_0} + cu - u^3 = 0$, where b and c are real numbers. Then the inequality (2) reads $\varepsilon^2 + \varepsilon b + c \leq 0$ for some $\varepsilon > 0$. (And observe that if $f(u) = cu - u^3$ the condition is on $c = \sup f'(u)$ but not on $\sup|f'(u)|$). This is equivalent to $4c \leq b^2$ and $b < \sqrt{b^2 - 4c}$. This has a physical interpretation if we think in equilibrium solutions of the reaction-diffusion equation with convection $u_t - bu_{x_0} = \Delta u + cu - u^3$ in Ω, where $-b$ means a downstream convection and $cu - u^3$ a nonlinear reaction.

Then, in order to fulfill H3, if $c < 0$ (pure dissipation) every convection is allowed, but if $c \geq 0$ (reaction) then a positive downstream convection is needed.

3. Global behaviour

Our first result connects our problem with the theory of infinite-dimensional dynamical systems.

THEOREM 2. *Let the hypotheses H1 to H5 be satisfied.*

i) *Suppose that the functions $a_{ij}(x)$ and $f(p,z,x)$ do not depend on x_0 (autonomous case). Let $u(x)$ be the (unique) solution of (1) for a given φ and $s_0 = 0$. Then $T(s)\varphi := u(s,\cdot)$ is a (nonlinear) C^0-semigroup in the Banach space $C_0^0(\overline{\Omega}')$ which is completely continuous for $s > 0$ and has a connected compact global attractor.*

ii) *Suppose that the functions $a_{ij}(x)$ and $f(p,z,x)$ are periodic in x_0 with period $\omega > 0$. Let $u^{s_0}(x)$ be the (unique) solution of (1) for a given φ. Then $\widetilde{T}(s)(s_0,\varphi) := \big((s_0+s)_{\mathrm{mod}\,\omega}, u^{s_0}(s_0+s,\cdot)\big)$ is a (nonlinear) C^0-semigroup in $(\mathbb{R}/\omega Z) \times C_0^0(\overline{\Omega}')$ which is completely continuous for $s > 0$ and has a connected compact global attractor.*

Finally let us give one sufficient condition for all the solutions of (1) to tend to zero as x_0 tends to infinity.

THEOREM 3. *Let the hypotheses H1 to H5 be satisfied and further that*

$$\nu_1 \sum_{i=0}^{n} \xi_i^2 \leq \sum_{i,j=0}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu_2 \sum_{i=0}^{n} \xi_i^2$$

for some $\nu_1, \nu_2 > 0$ and all $x \in \overline{\Omega}$ and $(\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, and also that

$$\left| \frac{\partial a_{ij}(x)}{\partial x_k} \right| \leq \nu_3$$

for some $\nu_3 > 0$ and all $x \in \overline{\Omega}$ and $i,j,k = 0,1,\ldots,n$. Suppose further that

i) *The function $\partial f/\partial p_0$ is uniformly bounded for $x \in \Omega$ and p and z bounded.*

ii) *The function f is of the form $f(0,z,x) = zC(z,x)$ where C satisfies that given any $M > 0$ there exists $c(M) < 0$ such that if $|z| < M$ then $C(z,x) \leq c(M)$ for all $x \in \Omega$.*

Then if $u(x)$ *is a solution of* (1) *it follows that* $\lim u(x) = 0$, *exponentially, as* x_0 *tends to infinity, uniformly on* $(x_1, x_2, \ldots, x_n)$.

REFERENCES

[1] CALSINA, A.—SOLÀ-MORALES, J.—VALÈNCIA, M.: *Bounded solutions of some nonlinear elliptic equations in cylindrical domains*, preprint, 1993.

[2] KIRCHGÄSSNER, K.: *Wave solutions of reversible systems and applications*, J. Differential Equations **45** (1982), 113–127.

[3] CHEN, X. Y.—MATANO, H.—VERÓN, L.: *Anisotropic singularities of solutions of nonlinear elliptic equations in* $\mathbb{R}^2$, J. Funct. Anal. **83** (1989), 50–97.

[4] MIELKE, A.: *Inertial Manifolds for elliptic problems in infinite cylinders*, preprint, Univ. of Stuttgart.

[5] CALSINA, A.—MORA, X.—SOLÀ-MORALES, J.: *The dynamical approach to elliptic problems in cylindrical domains, and a study of their parabolic singular limit*, J. Differential Equations **102** (1993), 244–304.

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