Peter Poláčik

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In: Pavol Brunovský and Milan Medveď (eds.): Equadiff 8, Czech - Slovak Conference on Differential Equations and Their Applications. Bratislava, August 24-28, 1993. Mathematical Institute, Slovak Academy of Sciences, Bratislava, 1994. Tatra Mountains Mathematical Publications, 4. pp. 179--185.

Persistent URL: http://dml.cz/dmlcz/700106

Tatra Mountains Math. Publ. 4 (1994), 179–185

TATRA MOUNTAINS
Mathematical Publications

REALIZATION OF THE DYNAMICS OF ODE'S IN SCALAR PARABOLIC PDE'S

PETER POLÁČIK

ABSTRACT. We consider scalar parabolic PDE's $u_t = \triangle u + f(x,u,\nabla u)$ on a bounded, at least two-dimensional domain. We are interested in ODE's that are realizable in PDE's of this form. We say of an ODE that it is realizable if its dynamics is equivalent to the dynamics on an invariant manifold of some PDE in the considered class. The main results state that all linear ODE's (in any dimension) are realizable, and any (nonlinear) ODE has an arbitrarily small realizable perturbation. We also state analogous results for periodically forced equations of the form $u_t = \triangle u + g(t,x,u)$.

Introduction

Consider a semilinear parabolic problem of the form

$$u_t = \triangle u + f(t,x,u,\nabla u), \quad x \in \Omega\,, \tag{1}$$

$$u|_{\partial\Omega} = 0\,, \tag{2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and f is of class C^1 in all variables. We always assume that f is periodic in t, or even independent of t. Problem (1), (2) then defines a local dynamical system with discrete or continuous time, depending on whether (1) is time-periodic or autonomous (see [He1]). The state space of this system is an appropriate Banach space of functions on $\bar{\Omega}$ satisfying (2). Varying the nonlinearity in the equation and the domain Ω, we thus obtain a class of dynamical systems, let us call it $\mathcal{D}$.

Due to the relevance of (1), (2) in applied sciences and perhaps also due to the relatively simple form of the equation, the systems of class $\mathcal{D}$ are frequently subject to investigation in research articles (see, e.g., the monographs [He1, Ha, Ba-V, Te] and references therein). In spite of that, not many general

AMS Subject Classification (1991): 35K57, 35B40.
Key words: scalar parabolic equations, realization of vector fields, invariant tori, chaos.

results are available on behavior of bounded trajectories of the dynamical system generated by (1), (2) if $N = \dim \Omega$ is greater than 1. On the other hand, the one-dimensional problems are very well understood. For example, there is a universal description (i.e., description valid for all C^1 nonlinearities) of the asymptotic behavior of bounded trajectories, namely convergence to a fixed point (see [Br-P-S], or [Ze, Ma] for the autonomous case). Other results establishing a special structure of the systems in $\mathcal{D}$ if $N = 1$ can be found in [An, He2, C-C-H, C-P].

The presence of general dynamical results, such as a description of behavior of all trajectories, in a class of dynamical system is, of course, highly desirable. They help very much in the study of any particular model in this class. Unfortunately, higher dimensional problems (1), (2) turn out to be very different from the one-dimensional problems. No universal description of the behavior of trajectories exists for $N \geq 2$. In this case class $\mathcal{D}$ is just too big. There seems to be no restriction on what kind of dynamics we can find, for example, when we look at finite-dimensional invariant manifolds of (1), (2).

In order to justify these statements, we present below results on the realization of ODE's in (1), (2). One of these results says that, given any ODE, one can find an arbitrarily small perturbation which has a realization in (1), (2). By the latter we mean that the perturbed ODE generates a dynamical system which is the same, in the sense of C^1 equivalence, as the dynamical system on an invariant manifold of some problem (1), (2). Using these results we easily show that chaos (shift dynamics) can be found in some problem (1), (2).

Another result stated below implies that (1), (2) do not generate low dimensional dynamics in general, not even in the limit set of a single trajectory. In fact the dimension of the ω-limit set of a trajectory can be arbitrarily large even if the dimension of Ω is required to be fixed (equal to 2, for example).

The possibility of complicated dynamics of (1), (2), that is established by our realization results, has to be taken into acount when a model of the form (1), (2) is analyzed. In applications, however, one often neglects exceptional phenomena and is interested in what happens typically (generically). It can be shown that a typical bounded solution of (1), (2) converges to a periodic orbit (of period possibly higher than the period of the equation). Thus trajectories with complicated behavior are always confined to relatively "small" sets in the state space (see [P-T1, P-T2]).

We first formulate the realization results for autonomous equations (Section 2). Then we consider the time-periodic case with a more restricted class of nonlinearities $f = f(t, x, u)$ (Section 3).

Below, Ω always stands for a smooth bounded domain in some $\mathbb{R}^N$. By $u(t, \cdot, u_0)$, with u_0 in an appropriate Banach space, we denote the solution of (1), (2), defined on the maximal time interval, that satisfies the initial condition $u(0, \cdot) = u_0$.

Autonomous problem

Fix any $N \geq 2$. Consider the problem

$$u_t = \Delta u + f(x, u, \nabla u), \quad x \in \Omega \subset \mathbb{R}^N, \tag{3}$$

$$u|_{\partial\Omega} = 0, \tag{4}$$

where $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is of class C^1. This problem is well-posed on a fractional power space associated with the Laplacian and the boundary condition. Specifically, we choose $L_p = L_p(\Omega)$ with a $p > N$ as the basic space. Then a fractional power space $X \subset W^{2,p} \cap W_0^{1,p}$ of the L_p-realization of the Laplacian can be chosen such that $X \hookrightarrow C^1(\bar{\Omega})$ (cf. [Hel]). By the theory of [Hel], (3), (4) defines a (local) semiflow on X. For any $u_0 \in X$, we set

$$S(t)u_0 = u(t, \cdot, u_0),$$

as long as the latter is defined. Now suppose $S(t)$ admits a finite-dimensional locally invariant manifold, that is, there is a submanifold $W \subset X$ with $\dim W < \infty$, such that for each $u_0 \in W$ one has $S(t)u_0 \in W$ for t in an interval $[0, t_0)$. Taking the restriction $S(t)|_W$ we obtain a semiflow on W. In constructions of invariant manifolds one usually obtains W as a graph over an open set B in a finite dimensional subspace of X. In that case the flow on W can be represented by an ODE. We are interested in an "inverse problem". What kind of ODE's can be obtained as such representations for problems of the form (3), (4). More specifically, let B be a unit ball in $\mathbb{R}^n$ for some n. Consider an ODE

$$\dot{y} = h(y), \quad y \in B, \tag{5}$$

where $h: \bar{B} \to \mathbb{R}^n$ is a C^1 function. We say that (5) *can be realized in* (3), (4) if there exist a domain $\Omega \subset \mathbb{R}^N$ and $f \in C^1(\bar{\Omega} \times \mathbb{R}^{N+1}, \mathbb{R})$ such that (3), (4) has a locally invariant manifold W and the flow of (3), (4) on W is C^1-equivalent to the flow of (5).

Recall that the C^1-equivalence requires that there be a C^1-diffeomorphism of B onto W which maps trajectories of (5) onto trajectories of $S(t)|_W$, preserving the orientation by time.

We can now formulate our main realization results for the autonomous problems.

THEOREM 1. *Let $N \geq 2$ and $n \geq 1$ be arbitrary. Then the set of C^1 functions $h: \bar{B} \to \mathbb{R}^n$ such that (5) can be realized in (3), (4) contains*

(i) *all linear functions, and*
(ii) *a dense subset of $C^1(\bar{B}, \mathbb{R}^n)$ endowed with the C^1 supremum norm.*

Note that this theorem implies that there are problems (3), (4) containing chaos as well as trajectories with high-dimensional ω-limit sets. Indeed, to establish existence of chaos, we show that an ODE (5), which has a transverse homoclinic orbit to a hyperbolic periodic orbit, can be realized in (3), (4). It is known that such equations (5) exist and that they "contain" shift dynamics (see, e.g., [Wi, Gu-H]). Of course, if such an equation is perturbed slightly in the C^1 topology then the new equation will still have a transverse homoclinic orbit. Statement (ii) allows us to choose such a small perturbation which can be realized in (3), (4).

To show that (3), (4) can have trajectories with ω-limit set of any dimension, we apply statement (i). Given any m, we choose a linear equation on $\mathbb{R}^{2m}$ with trajectories dense in an m-torus. By (i) it follows that such trajectories can be found for some problem (3), (4). We emphasize that we do not need to increase $N = \dim \Omega$, when we increase n. In particular, all this can be done with two-dimensional domains.

For the proof of Theorem 1 we refer the reader to [Po3]. Actually, in [Po3] only functions h satisfying $h(0) = 0$ are considered. Though this additional condition is not necessary for realization results it made the proof of [Po3] shorter, since results of the previous paper [Po1] were applicable.

For related results see also [Po2] and [Ry1]. In these two papers realizability of (5) for all sufficiently smooth functions functions $h\colon \mathbb{R}^n \to \mathbb{R}^n$ (not just for a dense set) is shown, but only under the restriction $n \leq N$ (which is a rather severe one if $N = 2$). In [Po2], the result has been proved for the Neumann problem. The method, an elementary one, does not apply to Dirichlet boundary condition. The method of [Ry1] is very different and it works for both these types of boundary conditions.

To conclude the section on autonomous problems, we would like to mention the papers [Ha2, Ha3, Ry2, Fa-M] and [Fi-P], where realization results for delay differential equations and for nonlocal parabolic equations in one space dimension, respectively, have been obtained. In fact, some ideas of [Fi-P] have been used in [Po3].

Periodically forced problem

Consider the problem

$$u_t = \Delta u + f(t,x,u), \quad x \in \Omega \subset \mathbb{R}^N, \tag{6}$$

$$u|_{\partial\Omega} = 0, \tag{7}$$

where $f\colon \mathbb{R}\times\bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a C^1 function periodic in t: $f(t+\tau,\cdot,\cdot) \equiv f(t,\cdot)$ for some $\tau > 0$. Again choose $p > N$. This time we can take $X = W_0^{1,p} \hookrightarrow C(\bar{\Omega})$ as the state space, i.e., the space where (6), (7) is well posed (see [Hel]). We consider the dynamical system on X generated by the period map of (6), (7). This is the map $F: u_0 \mapsto u(\tau, \cdot, u_0)$, defined for those initial conditions $u_0 \in X$ whose solutions exist up to $t = \tau$. Thus by a trajectory of u_0 we understand the trajectory with respect to F, that is, the sequence $F^n u_0$, $n = 0, 1, \dots$. Similarly, the ω-limit set of u_0, $\omega(u_0)$, refers to the limit set of the sequence $F^n u_0$, if the latter is defined for all n.

As for (3), (4), we want to show that (6), (7) can have complicated dynamics. Notice that, unlike for (3), (4), if f in (6) is independent of t then no chaos can occur. (In fact, the autonomous problem is gradient-like, the standard energy functional being the Lyapunov functional.)

The basic concept in this section is realization of period maps of time-periodic ODE's, which is analogous to the notion of realization of autonomous ODE's from the previous section. Let $h: S^1 \times B \to \mathbb{R}^n$ be a C^1 function. Here B is the unit ball in $\mathbb{R}^n$ and $S^1 = \mathbb{R}/\mathbb{Z}$ (thus $h(t, y)$ is periodic in t). Consider the ODE

$$\dot{y} = h(t, y), \quad y \in B. \tag{8}$$

The period map H of (8) is defined in a standard way. In particular, its domain, $D(H)$, consists of initial conditions of those solutions whose values on the whole period interval are contained in B.

We say that the period map H of (8) *can be realized* in (6), (7), if there exist $\Omega \subset \mathbb{R}^N$ and a C^1-function $f: \mathbb{R} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, periodic in the first variable, such that the period map F of (6), (7) has the following property: There is a C^1-imbedding $\varphi: B \to \varphi(B) = W \subset X$ such that for all $y_0 \in B$ one has $y_0 \in D(H)$ iff $F(\varphi(y_0))$ is defined and contained in W, and, in addition, if the latter holds then $F(\varphi(y_0)) = \varphi(H(y_0))$.

Note that in case $D(H) = B$ we can simply say that W is positively invariant under F (i.e., $F(W) \subset W$) and $F/_W$ is C^1 conjugate to H.

THEOREM 2. *Let $N \geq 2$ and $n \geq 1$ be arbitrary. The set of C^1 functions $h: S^1 \times \bar{B} \to \mathbb{R}^n$ such that the period map of (8) can be realized in (6), (7) contains*

(i) *all functions linear in y,*

and

(ii) *a dense subset of $C^1(S^1 \times \bar{B}, \mathbb{R}^n)$ endowed with the C^1 supremum norm.*

This theorem, similarly as Theorem 1 in the autonomous case, implies that chaos, as well as trajectories dense in an m-torus, m arbitrary, can be found in (3), (4). The arguments are quite analogous.

We remark that a special case of our result, existence of trajectories dense in a circle, has been proved before by Dancer [Da]. He has also proved this result for periodically forced reaction-diffusion on the circle. Under presence of convective terms, these one-dimensional problems have been later studied by Sandstede and Fiedler [Sa-F]. They showed realizability of any ODE (8) on $\mathbb{R}^2$.

The proof of Theorem 2 will appear in a forthcoming paper.

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Received December 6, 1993

Comenius University
Institute of Applied Mathematics
Mlynská dolina
842 15 Bratislava
SLOVAKIA

E-mail: polacik@fmph.uniba.sk