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A FINITE ELEMENT METHOD FOR THE COMPUTATION OF PARAMETRIC MINIMAL SURFACES

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ABSTRACT. We present a numerical method for the computation of discrete solutions of the Plateau Problem. This problem consists of the investigation of minimal surfaces bounded by a prescribed Jordan curve in space. The numerical method allows to compute unstable minimal surfaces with prescribed boundary. It is based on a Boundary Element Method for which asymptotic convergence was proved and which uses the Douglas Integral. Here we extend the BEM to a Finite Element Method for piecewise linear elements.

1. Introduction

Plateau's Problem has always served as a model problem for highly nonlinear problems in analysis and the calculus of variations. This article is thought to give an idea how this problem can be solved numerically using piecewise linear Finite Elements.

In [DH] the authors present a Boundary Element Method for the Plateau Problem and prove asymptotic convergence for the method. Here we want to show how this method can be extended to a Finite Element Method which is numerically more efficient.

The Plateau Problem consists of finding area minimizing surfaces of disk type which span a given Jordan curve in space. In simple configurations a solution of this problem can be found experimentally by soap film experiments. But since the topological type of the solution is prescribed and since one is interested in unstable solutions numerical computations are of special interest.

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2. Some Theoretical Background

2.1 The Classical Approach.

Let Γ be a Jordan curve in $\mathbb{R}^3$ and let $B = \{z = (x,y) \mid |z| < 1\} = \{re^{i\phi} |\ 0 \le r < 1,\ 0 \le \phi < 2\pi\}$ be the unit disk in $\mathbb{R}^2$. We look for surfaces $u\colon \overline{B} \to \mathbb{R}^3$ such that ∂B is mapped onto Γ in a monotone way and which are stationary for the area functional

$$A(u) = \int_B |u_x \times u_y| \tag{1}$$

with respect to the class of admissable functions

$$\mathcal{C}'(\Gamma) = \{u \in H^1(B)^3 \cap C^0(\partial B)^3 | u|_{\partial B}\colon \partial B \to \Gamma \quad \text{weakly monotone}\}. \tag{2}$$

A surface which is regular in the differential geometric sense and which minimizes area has mean curvature zero everywhere. This will be taken as a definition of a minimal surface. For twodimensional surfaces after conformal reparametrization this fact can be expressed as follows.

DEFINITION 1. The function $u \in \mathcal{C}'(\Gamma)$ *solves the Plateau Problem*, if u is harmonic and conformal, i.e., if on B

$$\Delta u = 0\,, \tag{3}$$

and

$$|u_x| = |u_y|\,, \quad u_x \cdot u_y = 0\,. \tag{4}$$

Since the area functional is invariant under arbitrary diffeomorphisms of the unit disk it is more convenient to work with Dirichlet's integral

$$D(u) = \frac{1}{2}\int_B |u_x|^2 + |u_y|^2, \tag{5}$$

for which we have that

$$A(u) \le D(u) \tag{6}$$

and equality holds iff u is conformally parametrized, i.e., (4) holds. Dirichlet's integral is conformally invariant, so that we shall have to factor out the conformal group by a suitable normalization. Classically this is done by a threepoint condition. One choses three fixed points $e^{i\phi_k}$, $(k = 1,2,3)$ on ∂B and three fixed points P_k, $(k = 1,2,3)$ on the curve Γ and imposes the condition that $u(e^{i\phi_k}) = P_k$, $(k = 1,2,3)$. Consequently one has to change the class of admissable functions to

$$\mathcal{C}(\Gamma) = \{u \in H^1(B)^3 \cap C^0(\partial B)^3 \mid u|_{\partial B}\colon \partial B \to \Gamma \quad \text{weakly monotone} \\ \text{and} \quad u(e^{i\phi_k}) = P_k, \quad k = 1,2,3\} \tag{7}$$

The classical existence result of Douglas and Rado states that for a rectifiable boundary curve there always exists a minimal surface (which in fact furnishes a minimizer for D and A).

THEOREM 2. *Let Γ be a rectifiable Jordan curve in $\mathbb{R}^3$. Then there exists a minimal surface $u \in \mathcal{C}(\Gamma)$.*

For complete information about the theory of minimal surfaces we refer to the books of Dierkes, Hildebrandt, Küster, Wohlrab [DHKW], J. C. C. Nitsche [N] and Struwe [St]. Here and in the following sections we will only mention some theoretical results which are important for the numerical treatment of parametric minimal surfaces. An important result for us will be the local regularity of minimal surfaces at the boundary proved by Hildebrandt, Nitsche, Jäger and Heinz [DHKW, 7.3, Thm. 1].

THEOREM 3. *Let u be a minimal surface which maps an open arc $\gamma \subset \partial B$ into an open portion $\Gamma' \subset \Gamma$ and assume that $\Gamma' \in C^{k,\alpha}$ for some $k \in \mathbb{N}$ and some $0 < \alpha < 1$. Then $u \in C^{k,\alpha}(B \cup \gamma)$.*

2.2 Staying within the Class of Harmonic Maps.

There is an elegant reformulation of the Plateau Problem [St] which will lead to a Boundary Element Method and to a Finite Element Method for the numerical solution. The idea is to work within the class of harmonic maps. Let

$$\gamma\colon \partial B \to \Gamma$$

be a smooth fixed parametrization of the given curve Γ. There is a one-one correspondence which associates with each boundary map $w\colon \partial B \to \mathbb{R}^3$ its unique harmonic extension

$$\Phi(w)\colon B \to \mathbb{R}^3\,,$$

specified by

$$\begin{aligned} \Delta\Phi(w) &= 0 \quad \text{in} \quad B\,, \\ \Phi(w) &= w \quad \text{on} \quad \partial B\,. \end{aligned} \tag{8}$$

DEFINITION 4. For $s \in C^0(\partial B, \partial B)$ let

$$E(s) = \frac{1}{2}\int_B |\nabla\Phi(\gamma \circ s)|^2\,.$$

Thus $E(s)$ is just the Dirichlet Energy of the harmonic extension of $\gamma \circ s$. $E(s)$ can be expressed directly in terms of the values of the function $\gamma \circ s$ by means of the Douglas Integral.

$$E(s) = \frac{1}{16\pi}\int_{\partial B}\int_{\partial B} \frac{|(\gamma \circ s)(\phi) - (\gamma \circ s)(\phi')|^2}{\sin^2(\frac{\phi-\phi'}{2})}\, d\phi\, d\phi'.$$

Here it is more convenient to use the normalization for $\xi(\phi) = s(\phi) - \phi$:

$$\int_0^{2\pi} \xi(\phi)\, d\phi = 0\,, \quad \int_0^{2\pi} \xi(\phi)\cos\phi\, d\phi = 0\,, \quad \int_0^{2\pi} \xi(\phi)\sin\phi\, d\phi = 0\,, \tag{9}$$

instead of the classical three point condition in (7).

Let H be the Hilbert space

$$H = \{\xi \colon \partial B \to \mathbb{R} \mid \xi \text{ satisfies (9) and } \|\xi\|_{H^{1/2}(\partial B)} < \infty\}$$

and define

$$T = H \cap C^0(\partial B)\,.$$

If γ is smooth enough then E is differentiable as a function

$$E \colon \mathcal{T} \to \mathbb{R},$$

where

$$\mathcal{T} = \{\mathrm{id} + \xi \mid \xi \in T\}\,.$$

THEOREM 5. *The function*

$$u = \Phi(\gamma \circ s)$$

is a solution of the Plateau Problem if and only if $s \in \mathcal{T}$ is monotone and stationary for E in the following sense:

$$\langle E'(s),\ \xi\rangle = 0 \qquad \forall \xi \in T\,. \tag{10}$$

The last condition now opens the field for new methods for the numerical solution of the Plateau Problem.

3. Numerical Methods for the Plateau Problem

Here we briefly review the numerical methods which where used for the practical solution of the Plateau Problem. We omit numerical methods for minimal graphs as well as methods for the computation of conformal maps.

The numerically most successful approach up to now uses Courant's function for polygonal boundary curves. Courant's function was introduced in order to characterize all minimal surfaces spanned by a polygonal boundary curve as critical points of a function of finitely many variables. This can also be seen as a first step for the numerical computation of minimal surfaces.

Let the Jordan curve Γ be a polygon with vertices a_j and segments Γ_j, $(j = 1, \ldots, n+3)$. For given $\tau \in T = \{\tau \in \mathbb{R}^{n+3} \mid 0 < \tau_1 < \cdots < \tau_{n+3} < 2\pi\}$ the boundary of the parameter domain B is subdivided into the arcs $\gamma_j = \{e^{i\phi} \mid \tau_j < \phi < \tau_{j+1}\}$, $(j = 1, \ldots, n+3)$. In order to factor out the conformal group we assume that three components of τ are prescribed, e.g., $\tau_{n+k} = (k+1)\pi/2$, $k = 1, 2, 3$, although this choice is not good for the numerical procedure.

One then minimizes Dirichlet's integral over

$$X_C(\tau) = \{u \in H^1(B)^3 \mid u(\gamma_j) \subset \Gamma_j, \quad j = 1, \ldots, n+3\} \tag{11}$$

and defines Courant's function d_C as

$$d_C(\tau) = \inf_{u \in X_C(\tau)} D(u) = D\big(u_0(\tau,\cdot)\big)\,. \tag{12}$$

This infimum is achieved by a unique function $u_0(\tau,\cdot)$ which thus is harmonic in B and satisfies the boundary condition $u_0(\tau,\gamma_j) \subset \Gamma_j$, $j = 1,\ldots,n+3$. Note that this is not a linear boundary condition because Γ_j is a segment but no straight line. C o u r a n t [Cou] proved that $d_C \in C^1$ and $u_0(\tau,\cdot)$ is a minimal surface if and only if $\nabla d_C(\tau) = 0$.

This idea was used by J a r a u s c h [J] to compute approximations of minimal surfaces using a finite dimensional subspace of $X_C(\tau)$ consisting of Finite Elements on the unit disk B which were bilinear with respect to polar coordinates. The grid is much finer than the subdivision of ∂B which is given by the vector τ. But each boundary point $e^{i\tau_k}$ has to be a grid point.

Jarausch's method was extended to partially free minimal surfaces and even more general variational problems by W o h l r a b in [Wo]. See also K ö l l n e r [K].

There is a serious drawback for this method. The grid has to move according to the free parameters τ and the grid has a singular point at the origin. For practical purposes the regularity of Courant's function d_C is insufficient.

Heinz proved that a little change in the definition of Courant's function d_C makes it an analytic function d_S in T, which is called Shiffman's function. If we denote by $\overline{\Gamma}_j$ the straight line which contains the segment Γ_j, then Dirichlet's integral is minimized over the set

$$X_S(\tau) = \big\{u \in H^1(B)^3 \mid u(\gamma_j) \subset \overline{\Gamma}_j, \quad j = 1,\ldots,n+3\big\} \tag{13}$$

and one defines Shiffman's function d_S as

$$d_S(\tau) = \inf_{u \in X_S(\tau)} D(u) = D\big(u_0(\tau,\cdot)\big)\,. \tag{14}$$

Note that now, for given τ, the boundary condition $u(\gamma_j) \subset \overline{\Gamma}_j$, $j = 1,\ldots,n{+}3$, is a linear boundary condition for the harmonic function $u_0(\tau,\cdot)$ although with different Dirichlet and Neumann boundary conditions on the different parts γ_j of the boundary of the unit disk B.

H i n z e [Hi1, Hi2] discretized the linear problem (14) using piecewise linear Finite Elements on a triangular grid in B assuming that $e^{i\tau_k}$ are boundary nodes of the grid. He proved convergence in the $H^1(B)$ norm for the approximation of (14).

We should mention that besides the problem with a moving grid there is an additional problem with the numerical methods using Courant's or Shiffman's functions. Since the boundary of the unit disk is mapped into a polygon, singularities arise at the points $e^{i\tau_k}$ which are mapped into the vertices a_k of the polygon. These singularities reduce the order of approximation for the "linear" problems (12) and (14).

Some work has been done to go the direct way to minimal surfaces, namely to minimize the area functional A over some discrete space. Of course any such numerical method leads to theoretical and numerical problems because of the invariance of the area functional under arbitrary diffeomorphisms. W a g n e r [Wa1], [Wa2] used the area functional to minimize area for polyhedra spanned by a boundary curve. The same approach was used by S t e i n m e t z [Ste] for more complicated problems involving minimal surfaces, especially partially free minimal surfaces. See also T s u c h i y a [T2,T1].

Mean curvature flow of surfaces is the gradient flow for the area functional. This is used by D z i u k [D] to compute stable minimal surfaces by a Finite Element Method using Finite Elements on surfaces. A somewhat similar idea with infinite time step is used by P i n k a l l and P o l t h i e r [PP] to compute minimal surfaces and their conjugates.

A public-domain program, "Evolver", which can obtain minimizers for many discrete functionals (including the discrete Area Functional), has been written by B r a k k e [Br]. A discussion and analysis is provided in the User's Manual.

Following the lines of the proof of R a d o and D o u g l a s, T s u c h i y a gives an existence proof for discrete minimal surfaces in [T2,T3] and a convergence proof of the discrete surfaces to a continuous solution in the $H^1(D)$-norm. This convergence can be arbitrarily slow because the author uses an indirect argument in connection with the Courant–Lebesgue Lemma and so cannot prove any order of convergence with respect to the grid size. Although the result of T s u c h i y a seems to be the first complete convergence result for the approximation of minimal surfaces, it is proved for minimizers only.

A numerical method for the computation of solutions of Plateau's Problem which one could call a Boundary Element Method was proposed by W i l s o n in [Wi] who used the Douglas Integral.

Since the difference between Dirichlet's integral and Area always is nonnegative and $D(u) - A(u) = 0$ only for minimal surfaces u, H u t c h i n s o n [Hu] minimizes this difference, which is called the conformal energy of the surface u. In some situations this has significant numerical advantages over minimizing the Dirichlet energy. In addition arbitrary, not necessarily stable, minimal surfaces can be found in this way by a minimization procedure.

4. A Boundary Element Method

In [DH] the authors used the ideas from Section 2.2 to formulate a BEM for the computation of semi discrete solutions of the Plateau Problem by constructing a suitable finite dimensional subspace of T and H. For this let $\mathcal{G}_h$ be a grid on $\partial B \cong \mathbb{R}/2\pi$ with grid size h and define

$$T_h = \left\{\xi_h \in C^0(\partial B, \mathbb{R}) \,|\, \xi_h \in P_1(I) \ \forall I \in \mathcal{G}_h,\ \xi_h \text{ satisfies } (9)\right\}, \tag{15}$$
$$\mathcal{T}_h = \{\mathrm{id}\} + T_h\,.$$

Then $\mathcal{T}_h$ is a finite dimensional affine subspace of the affine space $\mathcal{T}$ and the semi-discrete version of the functional E is the restriction of E to the discrete space:

$$E_h = E|_{\mathcal{T}_h}\,.$$

A function $s_h \in \mathcal{T}_h$ is called a *semi-discrete stationary point for* E if

$$\langle E_h'(s_h),\ \xi_h\rangle = 0 \qquad \forall \xi_h \in T_h\,. \tag{16}$$

The associated function $u_h = \Phi(\gamma \circ s_h)$ is called a semi-discrete minimal surface. Note that u_h is analytic in the interior of B, but of course only Hölder-continuous on $\overline{B}$.

For smooth γ the main result from [DH] is the following

THEOREM 6. *Let s_0 be a non-degenerate stationary point for E with associated minimal surface $u_0 = \Phi(\gamma \circ s_0)$. Then there exists an $h_0 > 0$ such that if $0 < h \le h_0$ then there is a unique semi-discrete stationary point $s_h \in \mathcal{T}_h$ such that*

$$\|s_h - s_0\|_{H^{1/2}(\partial B)} \le ch^{3/2} \quad \text{and} \quad \|s_h - s_0\|_{C^0(\partial B)} \le ch^{3/2}|\ln h|^{1/2}\,.$$

If $u_h = \Phi(\gamma \circ s_h)$ is the corresponding semi-discrete minimal surface, then

$$\|u_h - u_0\|_{H^1(B)} \le ch^{3/2} \quad \text{and} \quad \|u_h - u_0\|_{C^0(B)} \le ch^{3/2}|\ln h|^{1/2}\,.$$

For a detailed discussion of the non-degeneracy of a stationary point see [DH]. Roughly speaking this means that the kernel of the second derivative of E at s_0 is trivial. Here we only mention that this excludes minimal surfaces with branch points (in general). Note that unstable minimal surfaces are included in this Theorem.

5. A Finite Element Method

In contrast to the Boundary Element Method described in the previous section where we used the continuous harmonic extension of discrete boundary values we now use the discrete harmonic extension of discrete boundary values.

In the following we assume that

$$\mathcal{G}_h = \{T_k \mid k = 1, \ldots, nt\}$$

is a triangulation of the unit disk. The triangulation consists of triangles T and for every two triangles $T \neq T'$, $T \cap T'$ is an edge or a vertex of T and T'. h is the maximal diameter of a triangle. We also assume that all interior angles of all triangles $T \in \mathcal{G}_h$ are bounded from below by some uniform constant. Let

$$B_h = \bigcup_{k=1}^{nt} T_k\,.$$

We use the Finite–Element-space

$$X_h = \{v_h \in C^0(B_h, \mathbb{R}^3) \mid v_h \in P_1(T),\ T \in \mathcal{G}_h\} \tag{17}$$

where P_1 are the polynomials of degree ≤ 1. We should think of X_h as a continuation of the grid on the boundary (see the definition of T_h in (15) in the previous section) into the interior of ∂B. Without mentioning it explicitly we think of members of X_h to be continued constantly in radial direction to B.

We will use the following abbreviations:

$$\begin{aligned} nv &= \text{number of nodes of the triangulation } \mathcal{G}_h\,,\\ nb &= \text{number of boundary nodes}\,,\\ ni &= \text{number of interior nodes} = nv - nb\,,\\ nt &= \text{number of triangles}\,. \end{aligned}$$

By $x_j \quad (j = 1, \dots, nb,\ nb{+}1, \dots, nv)$ we denote the nodes of the triangulation.

For $g_h \in X_h$ denote by

$$\Phi_h(g_h)$$

the unique discrete harmonic extension of $g_h|_{\partial B_h}$ to B_h. Thus $\Phi_h(g_h)$ is uniquely specified by

$$\begin{aligned} &\Phi_h(g_h) \in X_h\,,\\ &\Phi_h(g_h) = g_h \quad \text{on} \quad \partial B_h\,, \end{aligned}$$

and

$$\int_{B_h} \nabla\Phi_h(g_h)\nabla\psi_h = 0 \tag{18}$$

for every $\psi_h \in X_h$ with $\psi_h = 0$ on ∂B_h.

For $u_h \in X_h$ we shall use the notations

$$u = (u_1, \dots, u_{nv})\,, \qquad u_j = u_h(x_j) \quad (j = 1, \dots, nv)\,. \tag{19}$$

By S we denote the stiffness matrix

$$S_{ij} = \int_{B_h} \nabla\psi_i\nabla\psi_j\,,$$

$(i, j = 1, \dots, nv)$ where ψ_j is the jth basis function of X_h, i.e., $\psi_j \in X_h$ and

$$\psi_j(x_k) = \delta_{jk}\,.$$

Thus

$$-S\colon \mathbb{R}^{nv} \to \mathbb{R}^{nv}$$

represents the discrete Laplace operator.

The discrete analogue of the Laplace operator with zero boundary conditions is given by the matrix $-S_0$ which is defined by

$$S_0 v = w \quad \text{iff} \quad Sv = w \quad \text{and} \quad v_j = 0 \quad \text{for} \quad j = 1, \dots, nb\,.$$

Let

$$\gamma = (\gamma^1, \gamma^2, \gamma^3)\colon \partial B \to \Gamma$$

be the fixed regular parametrization of the curve Γ. A discrete reparametrization $s_h \in T_h$, see (15), is represented by

$$s = (s_1, \dots, s_{nb}), \qquad s_j = s_h(x_j) \quad (j = 1, \dots, nb)\,, \tag{20}$$

respectively by $\big(\gamma(s_1), \dots, \gamma(s_{nb})\big)$. As fully discrete analogue of E resp. E_h we now take

$$E_{hh}(s_h) = \frac{1}{2} \int\limits_{B_h} |\nabla \Phi_h(I_h \gamma \circ s_h)|^2$$

where I_h is the piecewise linear interpolation operator on the boundary. In analogy to (16) a function $u_h = \Phi_h(I_h \gamma \circ s_h) \in X_h$ is called a discrete minimal surface if

$$\langle E'_{hh}(s_h), \xi_h \rangle = 0 \qquad \forall\, \xi_h \in T_h\,.$$

Lemma 7. *For given $s = (s_1, \dots, s_{nb})$ let*

$$u(s) = \big(u_1(s), \dots, u_{nv}(s)\big), \qquad u(s) = \big(u^1(s), u^2(s), u^3(s)\big)$$

be the discrete harmonic extension of $\gamma(s) = (\gamma(s_1), \dots, \gamma(s_{nb}), 0, \dots, 0) \in \mathbb{R}^{3nv}$, i.e.,

$$u(s) = -S_0^{-1} S \gamma(s) + \gamma(s)\,,$$

respectively

$$u^k(s) = -S_0^{-1} S \gamma^k(s) + \gamma^k(s), \quad k = 1, 2, 3\,. \tag{21}$$

The functional

$$E_{hh}(s) = \frac{1}{2} \int\limits_{B_h} |\nabla u_h(s)|^2$$

with $u_h(s) = \sum_{k=1}^{nv} u_j(s) \psi_j$ then can be written as

$$E_{hh}(s) = \frac{1}{2} \sum_{k=1}^{3} \big(S u^k(s),\ u^k(s)\big)\,. \tag{22}$$

Here and in the following $(\cdot\,,\cdot)$ stands for the euclidean scalar product in $\mathbb{R}^{nv}$, e_j is the j-th unit vector in $\mathbb{R}^{nv}$ and I the unit matrix in $\mathbb{R}^{nv}$.

The first derivatives of E_{hh} are given by

$$\frac{\partial E_{hh}}{\partial s_i}(s) = \sum_{k=1}^{3} \big(S u^k(s), (I - S_0^{-1} S) e_i\big) \gamma^{k\prime}(s_i) \tag{23}$$

$(i = 1, \dots, nb)$, *and the second derivatives are*

$$\begin{aligned}\frac{\partial^2 E_{hh}}{\partial s_i \partial s_j}(s) &= \big(S(I - S_0^{-1}S)e_i, (I - S_0^{-1}S)e_j\big) \sum_{k=1}^{3} \gamma^{k\prime}(s_i)\gamma^{k\prime}(s_j) \\ &\quad + \sum_{k=1}^{3} \big(Su^k(s), (I - S_0^{-1}S)e_j\big)\gamma^{k\prime\prime}(s_j)\delta_{ij}\,.\end{aligned} \tag{24}$$

$(i, j = 1, \dots, nb)$.

For the *proof* we only have to mention that for $k = 1, 2, 3$

$$\frac{\partial u^k}{\partial s_i}(s) = \gamma^{k\prime}(s_i)(I - S_0^{-1}S)e_i\,,$$

$(i = 1, \dots, nb)$ and

$$\frac{\partial^2 u^k}{\partial s_i \partial s_j}(s) = \delta_{ij}\gamma^{k\prime\prime}(s_j)(I - S_0^{-1}S)e_j\,,$$

$(i, j = 1, \dots, nb)$.

It is worth noting that the grid on B and consequently the stiffness matrix remains fixed during the computations. This is not only true for one special boundary curve but for any curve. It is then clear that the vectors

$$(I - S_0^{-1})e_j, \quad j = 1, \dots, nb\,,$$

should be computed once and never again for a given triangulation of the unit disk.

We add some test examples. First we compute the classical Enneper surface with parameter R which acts as a bifurcation parameter. It is well known that for $0 < R < 1$ there exists a unique minimizing Enneper surface, for $1 < R < \sqrt{3}$ there exist three solutions of Plateau's problem two stable minima and one unstable minimal surface. In this case we compute the unstable solution and calculate the experimental order of convergence between the linear interpolant of the smooth solution and the discrete solution. The boundary curve is given by

$$\begin{aligned}\gamma^1(s) &= R^3\cos(3s) + 4R^5\cos(5s)\,, \\ \gamma^2(s) &= R^3\sin(3s) - 4R^5\sin(5s)\,, \\ \gamma^3(s) &= -\sqrt{15}R^4\sin(4s)\,,\end{aligned}$$

for $s \in [0, 2\pi]$. The continuous solution is given by the harmonic continuation of this parametrization of the boundary curve. If e_h is the error between the continuous solution and the discrete solution, then for two successive grids with grid size h_1 and h_2 the experimental order of convergence is

$$\mathrm{eoc} = \ln\frac{e_{h_1}}{e_{h_2}} \Big/ \ln\frac{h_1}{h_2}\,.$$

Stable Enneper Surface (R=0.5)						
level	h	L^2-error	L^∞-error	H^1-error	eoc L^2	eoc H^1
1	0.7654	2.750 e-3	6.762 e-3	2.886 e-2		
2	0.3902	1.464 e-3	2.772 e-3	1.127 e-2	0.9	1.4
3	0.2102	4.435 e-4	9.195 e-4	3.493 e-3	1.9	1.9
4	0.1110	1.170 e-4	2.820 e-4	9.995 e-4	2.1	2.0
5	0.05687	2.950 e-5	8.329 e-5	2.763 e-4	2.1	1.9

Enneper Surface, R = 1.0						
level	h	L^2-error	L^∞-error	H^1-error	eoc L^2	eoc H^1
1	0.7654	9.808 e-2	0.2089	0.9343		
2	0.3902	7.118 e-3	1.139 e-2	5.225 e-2	3.9	4.3
3	0.2102	2.162 e-3	3.768 e-3	1.665 e-2	1.9	1.9
4	0.1110	5.723 e-4	1.151 e-3	4.792 e-3	2.1	2.0
5	0.05687	1.4474 e-4	3.3904 e-4	1.3265 e-3	2.1	1.9

Unstable Enneper Surface (R=1.5), pentagonal grid					
level	h	L^2-error	H^1-error	eoc L^2	eoc H^1
1	0.6641	7.650 e-2	0.3941	–	–
2	0.3320	1.538 e-2	0.1214	2.3	1.7
3	0.1843	4.670 e-3	3.788 e-2	2.0	2.0
4	0.09640	1.248 e-3	1.079 e-2	2.0	1.9
5	0.05686	4.103 e-4	3.688 e-3	2.1	2.0

The experimental results exhibit the well known superconvergence effects for linear elliptic equations at the nodes of the grid.

We do not expect convergence if the kernel of the second derivatives of E is nontrivial. The following example is made from the exact formula for a minimal surface $u_0 = \Phi(\gamma \circ s_0)$ with a branch point at the origin. In this case the kernel of $E''(s_0)$ is well known and we are able to subtract the singular part of the solution, i.e., project the solution onto the space orthogonal to the kernel.

Branch point (order=1, index=3)						
level	h	L^2-error	L^∞-error	H^1-error	eoc L^2	eoc H^1
1	0.7654	–	–	–	–	–
2	0.3902	7.151 e-3	9.712 e-3	3.455 e-2	–	–
3	0.2102	7.817 e-3	8.917 e-3	2.875 e-2	−0.1	0.3
4	0.1110	1.119 e-2	1.1709 e-2	3.9198 e-2	−0.6	−0.5

Branch point (order=1, index=3) Projection onto the regular part						
level	h	L^2-error	L^∞-error	H^1-error	eoc L^2	eoc H^1
1	0.7654	–	–	–	–	–
2	0.3902	3.761 e-3	5.230 e-3	1.878 e-2	–	–
3	0.2102	7.292 e-4	8.690 e-4	3.879 e-3	2.7	2.6
4	0.1110	9.881 e-5	1.4989 e-4	5.9771 e-4	3.1	2.9

In the following we show some examples of solutions of the Plateau Problem computed with the fully discrete Finite Element Method. The boundary curve can easily be seen from the graphics.

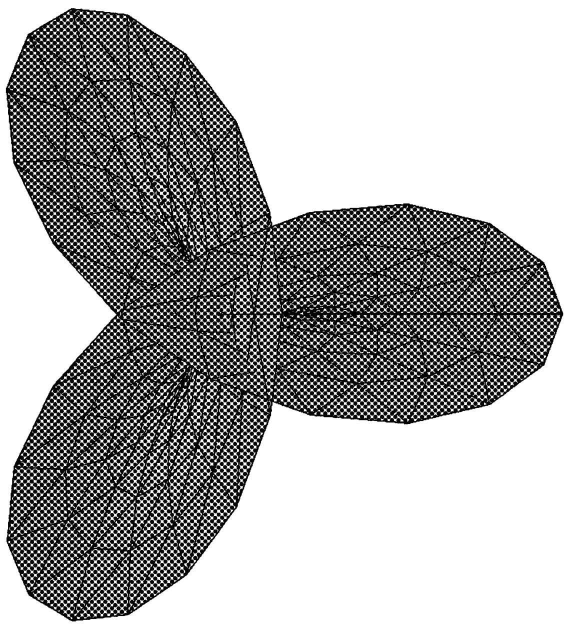

FIG. 1. : Discrete solution of the Plateau Problem with 545 nodes

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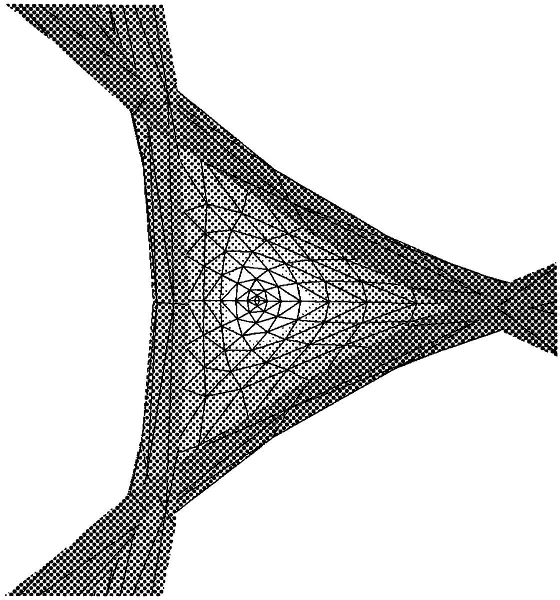

FIG. 2. : Close up and Cut of Fig. 1

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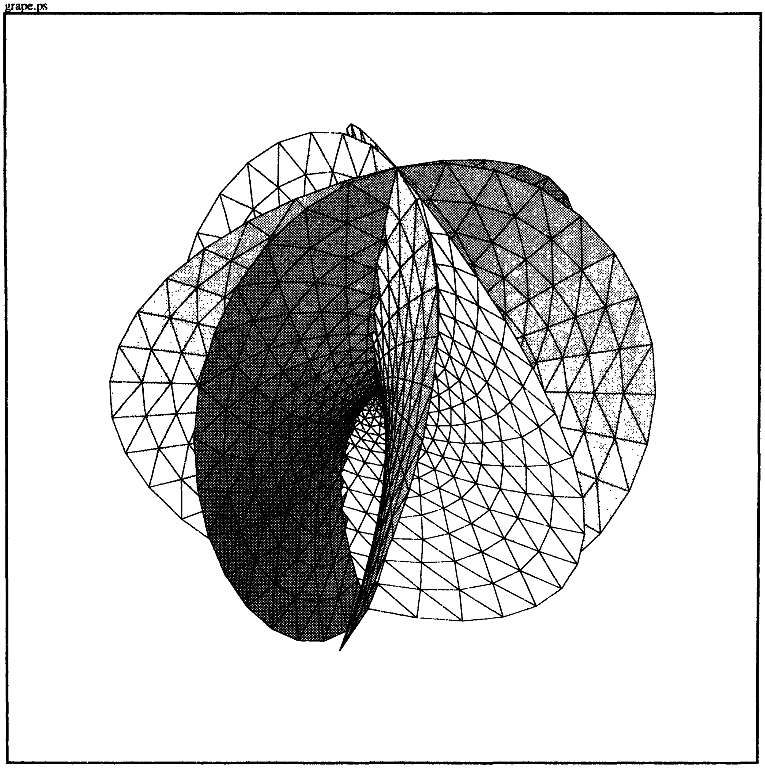

FIG. 3. : Solution of Plateau's Problem

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