ON TWO-SIDED DIFFERENCE METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS

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This paper contains some author's results [2] – [6].

§ 1. LINEAR PROBLEM S

Say, that a function $f(x) \in C_n$, $n \geqq 0$, if $f^{(n)}$ is continuous on [0, 1]. For the sake of simplicity we consider the two-point problem

$$Ly \equiv y'' + q(x)\,y = f(x), \quad 0 \leqq x \leqq 1, \quad y(0) = y(1) = 0, \tag{1.1}$$

where $q, f \in C_m$, $m \geqq 2$, are given functions taking arbitrary sign. It is known that for any solution of the problem (1.1) $y \in C_{m+2}$ holds and

$$y^{(\nu)} = \sum_{j=0}^{1} r_j^\nu(x)\,y^{(j)} + t^\nu(x), \quad 2 \leqq \nu \leqq m + 2, \tag{1.2}$$

where r_j^ν, t^ν are the known polynomials in $q^{(\mu)}, f^{(\mu)}$, $\mu = 0, 1, \ldots, \nu - 2$.

Denote $h = 1/N (N \geqq 2$ – natural$), f_k = f(kh)$, $L_h y_k = (y_{k+1} - 2y_k + y_{k-1})/h^2 + q_k y_k$, $\|f\| = \max\limits_{[0,1]} |f(x)|$, $\|f\|_h = \max\limits_{0 \leqq k \leqq N} |f_k|$.

Introduce the two-point difference problems

$$L_h \tilde{y}_k^{(0)} = f_k, \quad 0 < k < N, \quad \tilde{y}_{iN}^{(0)} = 0, \quad i = 0, 1, \tag{1.3}$$

$$L_h Z_k^j = 0, \quad 0 < k < N, \quad Z_{iN}^j = \delta_i^j, \quad i = 0, 1, \tag{1.4}$$

where $j = 0, 1$; δ_i^j is the Kronecker-symbol.

Theorem 1.1. *Suppose that for some fixed h the problem* (1.4) *has some solutions* Z^j, $j = 0, 1$. *Then for this h the problem* (1.4) *has no other solutions and the difference problem* (I.3) *is uniquely solvable for arbitrary* f_k, $k = 1, 2, \ldots, N - 1$.

Assume that the conditions of Theorem 1.1 hold and introduce the difference Green's function g_k^ν, $0 \leqq k \leqq N$, $1 \leqq \nu \leqq N - 1$, as the solution of the equations

$$L_h g_k^\nu = -\delta_k^\nu, \quad 0 < k < N, \quad g_{iN}^\nu = 0, \quad i = 0, 1. \tag{1.5}$$

We have

$$g_k^\nu = \begin{cases} C_\nu^1 Z_k^1, & 0 \leqq k \leqq \nu, \\ C_\nu^0 Z_k^0, & \nu \leqq k \leqq N, \end{cases} \tag{1.6}$$

where C_ν^0, C_ν^1, $\nu = 1, 2, \ldots, N - 1$, are constants defined as unique solution of the linear equations

$$C_\nu^0 Z_\nu^0 = C_\nu^1 Z_\nu^1, \quad L_h g_\nu^\nu = -1. \tag{1.7}$$

Let $i = 0, 1,$

$$D_h y_m = \frac{1}{2h} \cdot \begin{cases} (y_{m+1} - y_{m-1}), & 1 \leqq m \leqq N-1, \\ (-1)^i (4y_{i(N-2)+1} - 3y_{iN} - y_{i(N-4)+2}), & m = iN, \end{cases}$$

$\tilde{y}_k^{(1)} = D_h \tilde{y}_k^{(0)}$, $0 \leqq k \leqq N$, where $\tilde{y}^{(0)}$ is the solution of the problem (1.3). Assume that the problem (1.1) has a solution y. Then

$$\max_{1 \leqq k \leqq N-1} | L_h y_k - f_k | \leqq h^2 \| y^{(4)} \| / 12, \tag{1.8}$$

$$\| D_h y - y^{(1)} \|_h \leqq h^2 \| y^{(3)} \| / 3, \tag{1.9}$$

$$\| y^{(\tau)} \| \leqq \| \tilde{y}^{(\tau)} \|_h + \| \tilde{y}^{(\tau)} - y^{(\tau)} \|_h + h^2 \| y^{(\tau+2)} \| / 8, \tag{1.10}$$

$$\| y^{(\nu)} \| \leqq \sum_{j=0}^{1} R_j^\nu \| y^{(j)} \| + T^\nu, \qquad 2 \leqq \nu \leqq m+2, \tag{1.11}$$

where $\tau = 0$, 1 and $R_j^\nu \geqq \| r_j^\nu \|$, $T^\nu \geqq \| t^\nu \|$ are known numbers.

From (I.5)–(I.9) we have

$$\| \tilde{y}^{(\tau)} - y^{(\tau)} \|_h \leqq h^2 \sum_{\varkappa=3}^{4} B_\varkappa^\tau \| y^{(\varkappa)} \|, \tag{1.12}$$

where $\tau = 0, 1$, $B_3^\tau = \delta_1^\tau / 3$,

$$B_4^0 = \sum_{i=0}^{1} \| Z^i \|_h \sum_{\nu=1}^{N-1} | C_\nu^i | / 12,$$

$$B_4^1 = \sum_{i=0}^{1} \max_{1 \leqq \mu \leqq N} | Z_\mu^i - Z_{\mu-1}^i | \sum_{\nu=1}^{N-1} | C_\nu^i | / 6.$$

On the basis of (1.10)–(1.12)

$$\| y^{(\tau)} \| \leqq \| y^{(\tau)} \|_h + h^2 (\sum_{j=0}^{1} a_{\tau j} \| y^{(j)} \| + b_\tau), \tag{1.13}$$

where $\tau = 0, 1$, $a_{\tau j} = R_j^{2+\tau}/8 + \sum_{\varkappa=3}^{4} B_\varkappa^\tau R_j^\varkappa \geqq 0$, $b_\tau = T^{2+\tau}/8 + \sum_{\varkappa=3}^{4} B_\varkappa^\tau T^\varkappa \geqq 0$.

Theorem 1.2. (*Existence and uniqueness criterion for the solution of the differential problem*). *If for some fixed h the difference problem* (1.4) *has the solutions* Z^j, $j = 0, 1$, *and*

$$h^2 \max_{\tau=0,1} \sum_{j=0}^{1} a_{\tau j} < 1, \tag{1.14}$$

then the differential problem (1.1) *is uniquely solvable for any continuous* f.

Proof. If $f \equiv 0$, then $\| t^\nu \| = 0$, $2 \leqq \nu \leqq 4$, and by Theorem 1.1 $\| \tilde{y}^{(\tau)} \|_h = 0$, $\tau = 0, 1$. From here and (1.13), (1.14) taking $T^\nu = 0$, $2 \leqq \nu \leqq 4$, we easily deduce that any solution y of the problem (I.1) for $f \equiv 0$ has the norm $\| y \| = 0$, i.e. the problem (1.1) is uniquely solvable for any continuous f.

Theorem 1.3. *If the differential problem* (1.1) *is uniquely solvable, then there is such* $h^* > 0$ *that for any* $h = 1/N < h^*$ *the solutions* $Z^j, j = 0, 1,$ *of the difference problem* (1.4) *exist and the condition* (1.14) *holds.*

The proof of Theorem 1.3 is based on some results of G. M. Vainikko [1].

Let the conditions of Theorem 1.2 hold and let $\bar{Y}^\tau$, $\tau = 0, 1$ be a unique (by (1.14)) solution of two linear equations

$$\bar{Y}^i = \| \tilde{y}^{(i)} \|_h + h^2 (\sum_{j=0}^{1} a_{ij} \bar{Y}^j + b_i), \qquad i = 0, 1.$$

Let also

$$\bar{Y}^\nu = \sum_{j=0}^{1} R_j^\nu \bar{Y}^j + T^\nu, \qquad 2 \leqq \nu \leqq 4,$$

$$e^\tau = h^2 (\sum_{\varkappa=3}^{4} B_\varkappa^\tau \bar{Y}^\varkappa + \bar{Y}^{\tau+2}/8), \qquad \tau = 0, 1,$$

$$\tilde{Y}_\pm^\tau(x) = (1 - \delta_x) \tilde{y}_k^{(\tau)} + \delta_x \tilde{y}_{k+1}^{(\tau)} \pm e^\tau,$$

$$0 \leqq x \leqq 1, \quad k = \min\{N - 1, [x/h]\}, \quad \delta_x = x/h - k.$$

Theorem 1.4. *Let the solutions* Z^j, $j = 0, 1,$ *of the difference problem* (1.4) *exist and also* (1.14) *holds. Then*

$$\tilde{Y}_-^\tau(x) \leqq y^{(\tau)}(x) \leqq \tilde{Y}_+^\tau(x), \quad 0 \leqq x \leqq 1, \tag{1.15}$$
$$\| \tilde{Y}_+^\tau - \tilde{Y}_-^\tau \| = O(h^2), \quad 0 \leqq \bar{Y}^\tau - \| y^{(\tau)} \| = O(h^2),$$

where $\tau = 0, 1$ *and* y *is a unique solution of the problem* (1.1).

It is not difficult (see [4]) to obtain the two-sided approximation on [0, 1] of order $O(h^2)$ for $y^{(\nu)}$, $2 \leqq \nu \leqq m + 2$, using (1.2) and (1.15).

All the results formulated above are spread (see [5], § I) on the problem

$$\begin{aligned} Ly &\equiv y^{(2m)} + \sum_{k=0}^{2m-1} p^k(x)\, y^{(k)} = f(x), \qquad 0 \leqq x \leqq 1, \\ l_i y &\equiv \sum_{j=0}^{2m-1} \int_0^1 y^{(j)}(s)\, \mathrm{d}\mu_{ij}(s) = a_i, \qquad i = 1, 2, \ldots, 2m, \end{aligned} \tag{1.16}$$

where $m \geqq 1$; f, $p^k \in C_2$, μ_{ij} are given functions and a_i are given numbers. The functions μ_{ij} are piece-wise continuous and have continuous and bounded derivatives up to the third order with possible exception of finite number of points. No restrictions on $\| p^k \|$, $k = 0, 1, \ldots, 2m - 1$, are needed.

The many-point de la Vallée Poussin's problem is included in (1.16) as a special case.

In [2] – [5] we investigate the two-point problem for the linear equation of the second order with boundary conditions of the third kind when maximum principle is valid. The two-sided estimates for the error of the scheme of order $O(h^4)$ are constructed in [2], [3]. In [3] we give estimates of the error which are expressed explicitly by means of the coefficients of the equation and we consider also the method of obtaining point-wise estimates of the error which are practically more precise than uniform estimates. In [4], [5] we construct on [0, 1] the two-sided approximation for the solution and for the derivative of it with the order $O(h^2\omega(h) + h^3)$, where $\omega(t)$ is the sum of moduli of continuity of derivatives of order 2 of the coefficients of the equation.

§ 2. NONLINEAR AND SPECTRAL PROBLEMS

Consider the initial value problem

$$y'' = f(x, y), \quad 0 \leqq x \leqq 1, \quad y(0) = 0, \quad y'(0) = \lambda, \tag{2.1}$$

where $f(t_1, t_2)$ is a twice differentiable function on the rectangle $\mathscr{D}\{-H < t_1 < 1 + H, |t_2| < \bar{Y}\}$, $H > 0, \bar{Y} > 0$ and λ is a numerical parameter. Let

$$\bar{Y}^2(\lambda) \equiv F = \sup_{\mathscr{D}} |f|, \qquad F_i = \sup_{\mathscr{D}} \left|\frac{\partial f}{\partial t_i}\right|,$$

$$F_{ij} = \sup_{\mathscr{D}} \left|\frac{\partial^2 f}{\partial t_i \partial t_j}\right| < \infty, \qquad i, j = 1, 2,$$

$$\bar{Y}^3(\lambda) = F_1 + F_2(F + |\lambda|),$$

$$\bar{Y}^4(\lambda) = F_{11} + 2F_{12}(F + |\lambda|) + F_{22}(F + |\lambda|)^2 + FF_2$$

and let $w(\alpha, \beta, \gamma, a, b)$ be the value of the solution of the initial value problem

$$w'' = \alpha w' + \beta w + \gamma, w(0) = a, w'(0) = b$$

at the point $x = 1$, where α, β, γ are constants.

Denote

$$e^0(\lambda) = h^2 w(hF_2, F_2, \bar{Y}^4(\lambda)/12, 0, 11h\bar{Y}^4(\lambda)/72),$$

$$e^1(\lambda) = h^2 w(hF_2, F_2, 0, 5h\bar{Y}^4(\lambda)/72, \bar{Y}^4(\lambda)/12),$$

$$\dot{Y}^0 = w(0, F_2, 0, 0, 1), \quad \dot{Y}^1 = w(0, F_2, 0, 1, 0),$$

$$\ddot{Y}^0 = w(0, F_2, F_{22}(\dot{Y}^0)^2, 0, 0).$$

Let λ and h be given,

$$h = 1/N < 3H, \quad |\lambda| < 3\bar{Y}/h, \tag{2.2}$$

and values

$$\tilde{y}_0^{(0)}(\lambda) = 0, \quad v_0 = \lambda - hf(-h/3, -\lambda h/3)/2,$$

$$v_{k+1} = v_k + hf(kh, \tilde{y}_k^{(0)}(\lambda)),$$

$$\tilde{y}_{k+1}^{(0)}(\lambda) = \tilde{y}_k^{(0)}(\lambda) + hv_{k+1},$$

$$\tilde{y}_k^{(1)}(\lambda) = (v_k + v_{k+1})/2, \quad k = 0, 1, \dots, N,$$

exist. Let also $\tau = 0, 1$,

$$\tilde{Y}^{\tau}_{\pm}(x, \lambda) = (1 - \delta_x)\, \tilde{y}^{(\tau)}_k(\lambda) + \delta_x \tilde{y}^{(\tau)}_{k+1}(\lambda) \pm e^{\tau}(\lambda) \pm h^2 \bar{Y}^{\tau+2}(\lambda)/8,$$

$$0 \leqq x \leqq 1, \quad k = \min\{N - 1, \ [x/h]\}, \quad \delta_x = x/h - k.$$

Theorem 2.1. *If* (2.2) *holds for some* λ *and* h *and*

$$\| \tilde{y}^{(0)}(\lambda) \|_h + e^0(\lambda) + h(2F + |\lambda|) < \bar{Y}, \tag{2.3}$$

then the problem (2.1) *has for given* λ *the unique solution* y *on* $[0, 1]$. *Moreover,* $\| y \| < \bar{Y}$.

Theorem 2.2. *If for some* λ *there is a solution* y *of the problem* (2.1) *on* $[0, 1]$ *and* $\| y \| < \bar{Y}$, *then there is such* $h^* > 0$ *that for this* λ *and for any* $h = 1/N < h^*$ *the conditions* (2.2) *and* (2.3) *hold.*

Theorem 2.3. *If* (2.2), (2.3) *hold for given* λ *and* h, *then*

$$\tilde{Y}^{\tau}_{-}(x, \lambda) \leqq y^{(\tau)}(x, \lambda) \leqq \tilde{Y}^{\tau}_{+}(x, \lambda), \quad 0 \leqq x \leqq 1, \tag{2.4}$$

$$\| \tilde{Y}^{\tau}_{+} - \tilde{Y}^{\tau}_{-} \| = O(h^2), \quad \tau = 0, 1, \tag{2.5}$$

where $y(x, \lambda)$ *is the solution of the problem* (2.1).

Obviously, if for some λ the equality $y(1, \lambda) = 0$ holds, then $y(x, \lambda)$ being the solution (2.1) is the solution of the two two-point problems

$$y'' = f(x, y), \quad 0 \leqq x \leqq 1, \quad y(0) = y(1) = 0, \quad y'(0) = \lambda, \tag{2.6}$$

$$y'' = f(x, y), \quad 0 \leqq x \leqq 1, \quad y(0) = y(1) = 0, \tag{2.7}$$

too.

Theorem 2.4. *If* (2.2) *holds for* $\lambda = \lambda_i$ *and for some* h *and also*

$$|\tilde{y}^{(0)}_N(\lambda_i)| - e^0(\lambda_i) \geqq 0,$$

$$\| \tilde{y}^{(0)}(\lambda_i)\|_h + e^0(\lambda_i) + (\lambda_2 - \lambda_1)\, \dot{Y}^0/2 + h(2F + |\lambda_i|) < \bar{Y}$$

where $i = 1, 2$, $\lambda_2 > \lambda_1$ *and furthermore*

$$\tilde{y}^{(0)}_N(\lambda_1)\, \tilde{y}^{(0)}_N(\lambda_2) < 0, \tag{2.8}$$

then there exists $\lambda = \lambda_0 \in [\lambda_1, \lambda_2]$ *for which the problem* (2.6) *is solvable and also*

$$\dot{Y}^{\tau}_{-}(x, \lambda_i) \leqq y^{(\tau)}(x, \lambda_0) \leqq \dot{\tilde{Y}}^{\tau}_{+}(x, \lambda_i), \quad 0 \leqq x \leqq 1, \tag{2.9}$$

where $\tau = 0, 1$, $\dot{Y}^{\tau}_{\pm}(x, \lambda) = \tilde{Y}^{\tau}_{\pm}(x, \lambda) \pm (\lambda_2 - \lambda_1)\, \dot{Y}^{\tau}$.

Theorem 2.5. *If the conditions of the Theorem* 2.4 *hold and also*

$$\sum_{i=1}^{2} (|\tilde{y}^{(0)}_N(\lambda_i)| - e^0(\lambda_i)) - (\lambda_2 - \lambda_1)^2 \ddot{Y}^0 > 0,$$

then the λ_0 *indicated in Theorem* 2.4 *is unique on* $[\lambda_1, \lambda_2]$ *and also* $\mathrm{d}y(1, \lambda)/\mathrm{d}\lambda \neq 0$ *for all* $\lambda \in [\lambda_1, \lambda_2]$ *where* $y(x, \lambda)$ *is the solution of the problem* (2.1).

Theorem 2.6. *If all the conditions of Theorem* 2.4 *except for* (2.8) *hold and also*

$$| \tilde{y}_N^{(0)}(\lambda_i) | - e^0(\lambda_i) - (\lambda_2 - \lambda_1)^2 \ddot{Y}^0/8 > 0, \quad i = 1, 2,$$

then the problem (2.6) *is unsolvable for all* $\lambda \in [\lambda_1, \lambda_2]$.

Theorem 2.7. *Let the problem* (2.6) *have the solution* y *for* $\lambda = \lambda_*$, $\| y \| < \bar{Y}$ *and also we have* $\mathrm{d}y(1, \lambda_*)/\mathrm{d}\lambda \neq 0$ *for the solution of the problem* (2.1). *Then there exists such* $h^* > 0$ *that for any* $h = 1/N < h^*$ *there are* λ_i, $i = 1, 2$, *for which the conditions of the Theorems* 2.4, 2.5 *hold and also* $\lambda_1 < \lambda_* = \lambda_0 < \lambda_2$, $\lambda_2 - \lambda_1 = O(h^2)$,

$$\| \dot{Y}_+^\tau - \dot{Y}_-^\tau \| = O(h^2), \quad \tau = 0, 1, \quad i = 1, 2. \tag{2.10}$$

The search of the values λ_1, λ_2 indicated in Theorem 2.7 carried out by the method of division of λ in two and others by the help of Theorems 2.4–2.6.

Analogously, the two-sided difference method is constructed for the spectral problem

$$y'' + (\lambda r(x) + q(x)) y = 0, \quad 0 \leqq x \leqq 1, \quad y(0) = y(1) = 0, \tag{2.11}$$

where $r, q \in C_2$, $r(x) > 0$. Together with the problem (2.11) the initial value problem

$$y'' = -(\lambda r(x) + q(x)) y, \quad 0 \leqq x \leqq 1, \quad y(0) = 0, \quad y'(0) = 1 \tag{2.12}$$

and the two-point problem

$$y'' = -(\lambda r(x) + q(x)) y, \quad 0 \leqq x \leqq 1, \quad y(0) = y(1) = 0, \quad y'(0) = 1$$

are considered. For any eigenvalue λ^* of the problem (2.11) we have $\mathrm{d}y(1, \lambda^*)/\mathrm{d}\lambda \neq 0$, where $y(x, \lambda)$ is the solution of the problem (2.12). The presence of the two-sided approximations for the eigenvalue, for the spectral function and for its derivative on [0, 1] allows to find for sufficiently small h the number of the indispensable simple zeros on (0, 1) of the spectral function which differs by -1 from the index of the spectral function.

Some more general problems than (2.1), (2.7) and (2.11) for the equation of the second order are considered in [6].

Remark. If in practice the values of the function $f(x, y)$ are calculated with some errors with absolute values not exceeding δ_0 we put

$$e^0(\lambda) = h^2 w\left(hF_2, F_2, \frac{\bar{Y}^4(\lambda)}{12} + \frac{\delta_0}{h^2}, 0, \frac{11h\bar{Y}^4(\lambda)}{72} + \frac{3\delta_0}{2h}\right),$$

$$e^1(\lambda) = h^2 w\left(hF_2, F_2, 0, \frac{5h\bar{Y}^4(\lambda)}{72} + \frac{\delta_0}{2h}, \frac{\bar{Y}^4(\lambda)}{12} + \frac{\delta_0}{h^2}\right).$$

Then if δ_0 is fixed, Theorems 2.1, 2.3–2.6 except for the statement (2.5) hold and if $\delta_0 = 0(h^2)$ all the Theorems 2.1–2.7 remain completely valid.

Similarly it is possible to control the influence of other round-off errors.

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