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REGULARITY OF POTENTIALS AND REMOVABILITY OF SINGULARITIES OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

by JOSEF KRÁL

We shall be concerned with certain results dealing with removability of singularities of solutions of partial differential equations. Let $G \subset R^n$ be an open set and let $P(D)$ be a differential operator of the form

$$P(D) = \sum_{\alpha} a_\alpha D^\alpha \tag{1}$$

acting on distributions in G; here the sum extends over a finite number of multi-indices $\alpha = [\alpha_1, \ldots, \alpha_n]$ and we write, as usual, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_k = -i\dfrac{\partial}{\partial x_k}$. We shall always assume that the coefficients a_α occurring in (1) are infinitely differentiable functions in G or complex constants. Given a class $K(G)$ of distributions (or functions) in G, a set $F \subset G$ which is closed in G is termed removable with respect to $P(D)$ and $K(G)$ provided every $u \in K(G)$ satisfying $P(D)\, u = 0$ in $G \setminus F$ satisfies the same equation in the whole of G. Nowadays many results are known giving sufficient or necessary conditions for removability of a set F. Roughly speaking such results tell us that F is removable provided it is not very massive and describe the corresponding measure of massiveness in dependence on the operator $P(D)$ and the class $K(G)$ under consideration. We shall describe here several results of this sort dealing with the case when $K(G)$ consists of functions satisfying in G conditions of the Hölder type. Sharp result of this kind is due to L. Carleson who showed in [4] (see also [5]) that, for the Laplace operator Δ and the class $K^\beta(G)$ of functions satisfying in G locally the Hölder condition with exponent $\vartheta \in (0, 1)$, Hausdorff measures serve as a suitable measure of massiveness of the singular set. Je. P. Dolženko gave in [10], [11], also for the Laplace operator, conditions for removability that express in terms of the Hausdorff measures constructed with the help of the modulus of continuity of the function considered and its first order derivatives (see also [6], [9] dealing with removability of singularities of analytic functions of a complex variable). The sufficiency part of Carleson's theorem was extended by R. Harvey and J. Polking who proved in [17], among other things, the following very general theorem:

Let $P(D)$ be a differentiable operator (1) *of order m with infinitely differentiable coefficients in G. Then a set $F \subset G$ which is closed in G is removable with respect to $P(D)$ and $K^\beta(G)$ provided the $(n - m + \beta)$-dimensional Hausdorff measure of F equals zero.*

It is remarkable on this result that it gives a sufficient condition for removability of the singular set depending only on the order m of $P(D)$ and otherwise entirely independent of the type of the differential operator. It is natural to expect that result of this generality will not be sharp for all possible types of differential operators of order m and that, for obtaining necessary and sufficient conditions for removability of singular sets for special types of differential operators $P(D)$, it will be useful to consider special function spaces and also special measures of massiveness that are more closely connected with the operator $P(D)$. This was demonstrated on the example of the heat equation in [20] where results of the Carleson type were formulated with help of an anisotropic modification of the Hausdorff measure for functions satisfying the Hölder condition with exponent β in the space variables and exponent $\frac{1}{2}\beta$ in the time variable. We are now going to describe similar results for more general semielliptic differential operators. We shall always assume that $m = [m_1, \ldots, m_n]$ is a fixed n-tuple of positive integers satisfying

$$\sum_{k=1}^{n} \frac{1}{m_k} > 1$$

and, as usual, we denote for any multiindex $\alpha = [\alpha_1, \ldots, \alpha_n]$

$$|\alpha : m| = \sum_{k=1}^{n} \frac{\alpha_k}{m_k}.$$

We shall deal with differential operators of the form

$$P(D) = \sum_{|\alpha : m| \leqq 1} a_\alpha D^\alpha, \tag{3}$$

where α_α are infinitely differentiable functions in an open set $G \subset R^n$. Given a constant γ satisfying the inequalities

$$0 < \gamma < (\max_k m_k)^{-1} \tag{4}$$

we shall denote by $K_m^\gamma(G)$ the class of all functions u satisfying in G locally the condition

$$|u(x) - u(y)| = O(\sum_{k=1}^{n} |x_k - y_k|^{m_k \gamma}) \quad \text{as} \quad |x - y| \to 0+$$

(by this we mean precisely that for any compact $K \subset G$ there is a constant c_K such that $c_K \sum_{k=1}^{n} |x_k - y_k|^{m_k \gamma} \geqq |u(x) - u(y)|$ whenever $x, y \in K$). In order to be able to formulate conditions on removable singularities with respect to the class $K_m^\gamma(G)$ we are now going to define the anisotropic Hausdorff measures of type m. Let $r > 0$. By a distinguished parallelepiped of type m we mean any Cartesian product of the form

$$K = \mathop{\mathsf{X}}_{k=1}^{n} I_k,$$

where I_k is a one-dimensional interval of length $r^{\frac{1}{m_k}}$, $k = 1, \ldots, n$; the corresponding number r will be denoted by $|K|$ and will be called the order of the distinguished parallelepiped K of type m. It should be noted here that in connection with removability of singularities of solutions of partial differential equations distinguished parallelepipeds of type m were considered by W. LITTMAN [21] who used them for constructing measures analoguous to the Minkowskian content. We shall employ distinguished parallelepipeds of type m for defining anisotropic Hausdorff measures. Given ϱ, $\varepsilon > 0$ we let for any $M \subset R^n$

$$\mathscr{H}_m^{\varrho,\varepsilon}(M) = \inf \sum_j |K_j|^\varrho,$$

where the greatest lower bound on the right-hand side is taken over all sequences $\{K_j\}_j$ formed by distinguished parallelepipeds K_j of type m such that

$$M \subset \bigcup_j K_j \quad \text{and} \quad |K_j| < \varepsilon \quad \text{for all } j$$

The limit

$$\mathscr{H}_m^\varrho(M) = \lim_{\varepsilon \to 0+} \mathscr{H}_m^{\varrho,\varepsilon}(M)$$

will be termed the anisotropic ϱ-dimensional Hausdorff measure of type m. This is a Carathéodory outer measure whose restriction to the σ-algebra of measurable sets generates by the standard procedure a measure which will also be denoted by $\mathscr{H}_m^\varrho$. Now we are in position to formulate the following

Theorem 1. *Let $P(D)$ be a differential operator of the form* (3) *with infinitely differentiable coefficients in an open set $G \subset R^n$ and let $F \subset G$ be closed in G. If γ is a constant satisfying* (4), *ϱ is defined by*

$$\varrho = \gamma - 1 + \sum_{k=1}^{n} \frac{1}{m_k} \tag{5}$$

and $\mathscr{H}_m^\varrho(F) < \infty$, *then for every $u \in K_m^\gamma(G)$ satisfying $P(D)\,u = 0$ in $G \setminus F$ there is a locally bounded Baire function g_u in G vanishing on $G \setminus F$ such that in G the equality*

$$P(D)\,u = g_u \mathscr{H}_m^\varrho \tag{6}$$

holds in the sense of distribution theory (as usual, the right-hand side denotes the Radon measure defined by

$$\varphi \to \int_G \varphi(x)\, g_u(x)\, \mathrm{d}\mathscr{H}_m^\varrho(x)$$

on the class of continuous functions φ with compact support in G).

This result implies, in particular, the following

Corollary 1. *Let us keep the notation introduced in theorem* 1. *Then every F with $\mathscr{H}_m^\varrho(F) = 0$ is removable with respect to $P(D)$ and $K_m^\gamma(G)$.*

Indeed, for such F the measure occurring on the right-hand side of (6) becomes trivial ($=0$), because g_u has support contained in F.

If we apply this result to the heat conduction operator in R^n written in the form

$$iD_1 + D_2^2 + \dots + D_n^2$$

(so that the time variable is written on the first place), we have $m_1 = 1$, $m_2 = \dots = m_n = 2$ and the critical order of the Hausdorff measure corresponding to the Hölder exponent $\gamma = \frac{1}{2}\beta\,(0 < \beta < 1)$ equals $\frac{1}{2}\beta - 1 + 1 + \frac{1}{2}(n-1) = \frac{1}{2}(\beta + n - 1)$ so that $\mathscr{H}^{\frac{1}{2}(\beta+n-1)}(F) = 0$ is a sufficient condition for removability of the closed singular set $F \subset G$ with respect to the solutions of the heat equation in $G \setminus F$ satisfying in G locally the Hölder condition with exponent β in the space variables and exponent $\frac{1}{2}\beta$ in the time variable. It is known (see [20]) that this condition is also necessary.

If we consider, as another example, the Laplace operator $\Delta = -(D_1^2 + \dots + D_n^2)$, we have $m_1 = \dots = m_n = 2$ and the critical order ϱ of the Hausdorff measure corresponding to the class $K_m^\gamma(G)$, where $\gamma = \frac{1}{2}\beta$ with $0 < \beta < 1$, equals $\varrho = \frac{1}{2}\beta - 1 + \frac{1}{2}n = \frac{1}{2}(\beta + n - 2)$. Since now $m = [2, \dots, 2]$, $\mathscr{H}_m^\varrho(F) = 0$ means just the same as $\mathscr{H}^{\beta+n-2}(F) = 0$, where $\mathscr{H}^r$ denotes the ordinary r-dimensional Hausdorff measure; we have thus arrived, quite expectedly, at one part of the well-known result of L. Carleson asserting that $\mathscr{H}^{\beta+n-2}(F) = 0$ is necessary and sufficient for removability of F with respect to the Laplace operator and the class of functions satisfying the Hölder condition with exponent β.

In the above two examples the condition obtained from Corollary 1 was not only sufficient but also necessary for removability of the singular set. We may thus naturally ask whether the condition presented in Corollary 1 is also necessary for a wider class of differential operators including the Laplace operator and the heat conduction operator. The answer is affirmative for semielliptic differential operators with constant coefficients. From now on we shall consider the operators of the form (3) with constant complex coefficients. Let us recall that such an operator is called semielliptic provided the polynomial

$$P_m(x) = \sum_{|\alpha : m| = 1} a_\alpha x^\alpha$$

(where, as usual, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$) has no real roots except zero vector:

$$(x \in R^n, P_m(x) = 0) \Rightarrow x = 0\,.$$

This important class of operators includes the elliptic operators (for which $m_1 = \dots = m_n$) with the Laplace operator as a typical example and also operators parabolic in the sense of Petrovskij (for which $m_2 = \dots = m_n = pm_1$) with the heat conduction operator as a typical example (see [18], [29]). Assuming that $P(D)$ is semielliptic we shall denote by E the corresponding fundamental solution satisfying the equation $P(D)\,E = \delta$, where δ is the Dirac measure. $P(D)$ being hypoelliptic, E represents an infinitely differentiable function on $R^n \setminus \{0\}$. Given a measure μ with compact support we denote by $E\mu = E * \mu$ the convolution of E and μ. We shall be engaged with regularity of the potentials $E\mu$. More precisely, we are interested in conditions on a compact set $K \subset R^n$ guaranteeing the existence of a measure μ supported by K such that $E\mu \in K_m^\gamma(R^n)$ and $\mu(K) > 0$. Such conditions are known for certain concrete kernels in potential theory. For Newtonian kernel (where $m_1 = \dots = m_n$) they were established by L. CARLESON [4], the corresponding results for Riesz's kernels were proved by H. WALLIN [30] and for fundamental solution of the heat equation (where $m_2 = \dots = m_n = 2m_1$) they were presented in [20]. Although now no concrete representation of the kernel E is available, one can establish sharp estimates of the growth of E and its derivatives near the origin which permit the proof of the following

Theorem 2. *Let $K \subset R^n$ be a compact set and let $P(D)$ be a semielliptic differential operator (of the form (3)) with constant coefficients (so that m_k is just the degree of the polynomial $P(x) = \sum_{|\alpha : m| \leqq 1} a_\alpha x^\alpha$ with respect to the variable x_k, $k = 1 \dots, n$). If γ satisfies (4) and ϱ is given by (5), then $\mathscr{H}_m^\varrho(K) > 0$ is a necessary and sufficient condition for the existence of a measure μ with support* spt $\mu \subset K$ *such that $E\,\mu \in K_m^\gamma(R^n)$ and $\mu(K) > 0$.*

In combination with Corollary 1 this theorem gives us the following result:

Corollary 2. *Let $P(D)$, m, γ and ϱ have the same meaning as in Theorem 2. Let $G \subset R^n$ be an open set and let $F \subset G$ be closed in G. Then F is removable with respect to $P(D)$ and $K_m^\gamma(G)$ if and only if $\mathscr{H}_m^\varrho(F) = 0$.*

Indeed, $\mathscr{H}_m^\varrho(F) > 0$ implies that $\mathscr{H}_m^\varrho(K) > 0$ for suitable compact $K \subset F$ on which one can, by theorem 2, distribute a non-trivial measure μ with $E\mu \in K_m^\gamma(R^n)$. Since $P(D)\,E\mu = (P(D)\,E)_* \,\mu \doteq \mu$ this means that $P(D)\,E\mu = 0$ on $G \setminus F$ but is F not removable for $E\mu$.

As remarked in the beginning, there is a variety of results dealing with removability of singularities of solutions of partial differential equations. The space of this note being limited we cannot describe results dealing with other function spaces like Sobolev spaces, measures of massiveness like diverse capacities, and equations with more general coefficients, non-linear equations as well as equations which are not understood in the sense of distribution theory. We include a bibliography where further information may be found.

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