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Criteria for almost periodicity and some applications to differential equations

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CRITERIA FOR ALMOST PERIODICITY AND SOME APPLICATIONS TO DIFFERENTIAL EQUATIONS

by V. LOVICAR

1. Let $C(R^n)$ denote the linear space of uniformly continuous bounded functions on the set R of reals with values in R^n. On the set $C(R^n)$ we have a topology σ_1 given by supremum norm, and a topology σ_2 given by the metric ϱ:

$$\varrho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x - y)}{1 + d_n(x - y)},$$

where $d_n(x - y) = \sup_{t \in \langle -n, n \rangle} | x(t) - y(t) |$.

For $x \in C(R^n)$ let $P(x)$ denote the set $(p_t x; t \in R)$ of translates of $x (p_t x(s) = x(t + s)$ for $s \in R)$. Further we set $H(x) = cl_{\sigma_2} P(x)$. For any $x \in C(R^n)$ the set $H(x)$ is compact in the topology σ_2.

A function $x \in C(R^n)$ is called minimal if $H(y) = H(x)$ for any $y \in H(x)$. For any $x \in C(R^n)$ there exists $y \in H(x)$ which is minimal. If $x \in C(R^n)$ is minimal, then for any $\varepsilon > 0$ the set $S = (t \in R; \varrho(p_t x - x) \leqq \varepsilon)$ is relatively dense in R (this means that there exists $1 > 0$ such that any interval from R of the legth 1 has nonvoid intersection with S).

Theorem 1. Let $x \in C(R^n)$ be minimal and let $H(x)$ be separable in the supremum norm. Then x is almost periodic.

Theorem 1 follows from the following topological theorem, which in fact is due to Gelfand:

Theorem. Let X be a linear space on which the topologies σ_1 and σ_2 are defined such that in both of them X is a linear topological space. Let further $0 \neq M \subset X$ and let τ_j be the topology on M induced by the topology σ_j on X $(j = 1, 2)$.

Let the following assumptions be fulfilled:

1. There exists a countable fundamental system $(U_n; n \in N)$ of σ_1-neighbourhoods of $0 \in X$ such that $cl_{\sigma_1} U_n$ are σ_2-closed;
2. (M, τ_1) is separable;
3. (M, τ_2) is a compact Hausdorff space.

Then there exists $M_0 \subset M$ such that $cl_{\tau_2} M_0 = M$ and the identical mapping from (M, τ_2) to (M, τ_1) is continuous on the set M_0.

From the above theorem we also easily obtain

Theorem 2. Let B be a semicomplete (i.e. sequentially weakly complete) Banach space and x a weakly almost periodic function on R with values in B. Then for any $\varepsilon > 0$ there exists a relatively dense set M in R such that diam $(x(M)) = \sup_{t,s \in M} | x(t) - x(s) |_B \leqq \varepsilon$.

2. Let us consider the equation

$$x' = Ax \tag{1}$$

in a Banach space B, where A is a linear operator in B. We suppose that

$$\begin{aligned} \overline{D(A)} &= B, \\ \overline{D(A^*)} &= B^*. \end{aligned} \tag{2}$$

By a solution of (1) we mean a continuous function x on R with values in B such that for any $x^* \in D(A^*)$ and $f \in C_0^\infty(R)$ it holds

$$\int_{-\infty}^{+\infty} ((x(t), x^*) f'(t) + (x(t), A^* x^*) f(t)) \, \mathrm{d}t = 0.$$

Theorem 3. Let B be a complex Banach space and let A be a linear operator in B which fulfils the assumption (2) and such that $\sigma(A) \cap \mathrm{i}R$ does not contain any perfect subset. Then any bounded solution of (1) is weakly almost periodic.

For the proof of Theorem 3 see [4].

Theorem 4. Let B be a semicomplete complex Banach space and let A be a linear operator in B, which generates bounded semigroup of operators in B and such that $\sigma(A) \cap \mathrm{i}R$ does not contain any perfect subset. Then any bounded solution of (1) is almost periodic.

This theorem follows from Theorems 2 and 3.

3. As an easy consequence of the above theorem we have

Theorem 5. Let B be a semicomplete complex Banach space, let A be a generator of the bounded group of operators in B and let $\sigma(A)$ contain no perfect subset. Then the set of eigenvectors of A is total in B.

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